On the Uniqueness and Multiplicity of positive solutions to an elliptic spectral problem with concave and convex nonlinearity

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Abstract

Population models with a diffusive spread of a population following a nonlinear growth pattern have a long history. Selecting, for the growth, a mixture of a concave nonlinearity at low population densities and a convex one at high densities, in the late 1970s and early 1980s a number of researchers have developed powerful methods of partial differential equations and functional analysis to treat such problems assuming a *uniform, space-idependent* growth throughout the domain of the habitat. A typical result was the possibility of *two positive equilibria* of a lower and a higher population density, pointwise ordered throughout the habitat.

In our presentation we will first show the uniqueness of a positive solution for the linear case by showing a kind of "strict convexity" of the corresponding 1-homogeneous functional obtained by the simple substitution $v = u^2$ which turns a well-known quadratic functional with the variable u into the corresponding 1-homogeneous functional with the new variable $v = u^2$. We will demonstrate on a simple case that the new funcional still remains "strictly convex" in the positive function $v(x)$.

If the time still permits, we will construct a pair of positive equilibria in a model with *space-dependent growth*: concave in one subdomain and convex in the other one, linear on the boundary (or region) between the two subdomains.

We will discuss the question of ***existence*** and ***multiplicity*** of *positive solutions* to the semilinear elliptic Dirichlet problem

$$(0.1) \qquad -\Delta u = \lambda\, u(x)^{q(x)-1} + f(x, u(x)) \quad \text{for } x \in \Omega\,; \qquad u = 0 \quad \text{on } \partial\Omega\,.$$

Our main contribution is a method how to handle the interplay between *convex* and *concave* nonlinearities in two disjoint nonempty open subsets of a domain Ω (connected in $\mathbb{R}^N$), as opposed to the classical works assuming a nonlinearity $f(s)$ being *concave* for small values of $s \in \mathbb{R}_+$ and *convex* for large $s \in \mathbb{R}_+$, uniformly in Ω.

Running head: A semilinear elliptic problem with a convex/concave nonlinearity

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spectral parameter and critical values;
space-dependent exponent; convex / concave nonlinearity;
positivity and Hopf's boundary point lemma

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