Computational finite element methods

03 September, 2024 August–November Semester

Exercise sheet 1

1. Consider the reaction-diffusion equation

$$-\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = f$$

where f is a continuous function. In the class, the Lagrange P^1 FEM was implemented only with the diffusion term. In this question, we shall attempt to extend this to the reaction term y.

(a) Write a variational form for the differential equation over $V = \{y \in C^1(0,1) : y(0) = 0 = y(1)\}$.

(b) *Construction of the local mass matrix.* Note that in the variation form the term

$$\int_0^1 y\varphi \,\mathrm{d}x \tag{0.1}$$

appears, where $\varphi \in V$ is a test function. Recall that the local stiffness corresponding to the term $\int_0^1 y'\varphi' \,\mathrm{d}x$ is

$$\begin{pmatrix} \int_{x_j}^{x_{j+1}} \frac{\mathrm{d}\varphi_L}{\mathrm{d}x}\frac{\mathrm{d}\varphi_L}{\mathrm{d}x}\,\mathrm{d}x & \int_{x_j}^{x_{j+1}} \frac{\mathrm{d}\varphi_R}{\mathrm{d}x}\frac{\mathrm{d}\varphi_L}{\mathrm{d}x}\,\mathrm{d}x \\ \int_{x_j}^{x_{j+1}} \frac{\mathrm{d}\varphi_L}{\mathrm{d}x}\frac{\mathrm{d}\varphi_R}{\mathrm{d}x}\,\mathrm{d}x & \int_{x_j}^{x_{j+1}} \frac{\mathrm{d}\varphi_R}{\mathrm{d}x}\frac{\mathrm{d}\varphi_R}{\mathrm{d}x}\,\mathrm{d}x \end{pmatrix}.$$

Could you use this and (0.1) to construct the local stiffness matrix for (0.1)?

(c) Design and implement a Lagrange P^1 FEM for the reaction-diffusion problem.

2. (Conservative form) Consider the ordinary differential equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(a(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) = f$$

where f, a are continuous functions. Moreover, $0 < m \leq a(x) \leq M$ for some $m, M \in \mathbb{R}^+$. Design and implement the Lagrange P^1 FEM for this problem with homogenenous boundary conditions.

Hint. Use $\int_{x_j}^{x_{j+1}} a(x)f(x)\,\mathrm{d}x \approx a((x_j + x_{j+1})/2)\int_{x_j}^{x_{j+1}} f(x)\,\mathrm{d}x$.