08 January 2026 12:27

Review of Calculus.

Definition. 1.1 (Continuous function). Let $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, where Ω is an open set. Then the function f is continuous at $\vec{a} \in \Omega$ if for a given $\varepsilon > 0$, $\exists$ a corresponding $\delta > 0$ such that

whenever $\|\vec{x} - \vec{a}\|_{\mathbb{R}^n} < \delta$ and $\vec{x} \in \Omega$

then $\|f(\vec{x}) - f(\vec{a})\|_{\mathbb{R}^m} < \varepsilon$.

Remark. 1.2 If f is continuous at all $\vec{x} \in \Omega$, then f is said to be continuous on Ω.

Remark 1.3. $\|\cdot\|_{\mathbb{R}^d}$ is a norm $\mathbb{R}^d$. Several norms are possible in $\mathbb{R}^d$. We shall often work with the Euclidean norm defined by

$$\|\vec{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

where $\vec{x} = (x_1, x_2, \cdots, x_n)$. Note that all norms in $\mathbb{R}^d$ are equivalent. i.e, $\exists$ constants C_1 and C_2 such that

$$C_1 \|\cdot\|_A \leq \|\cdot\|_B \leq C_2 \|\cdot\|_A$$

for every pair of norms $\|\cdot\|_A$ and $\|\cdot\|_B$ on $\mathbb{R}^d$. The constants C_1 and C_2 are independent of the points in $\mathbb{R}^d$. Therefore therefore continuity WRT one norm implies continuity WRT all other norms.

Theorem 1.4 (Sequential Continuity) A function $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, where Ω is an open set is continuous at $\vec{a} \in \Omega$, iff for any sequence $(\vec{x}_n)_{n \geq 1}$ such that $\vec{x}_n \to \vec{a}$, $f(\vec{x}_n) \to f(\vec{a})$.

Proof. Assume that $(\vec{x}_n)_{n \geq 1} \subseteq \Omega$ and $\vec{x}_n \to \vec{a}$, and f is continuous at $\vec{a}$. Continuity assures that for any $\varepsilon > 0$, $\exists$ corresponding $\delta > 0$ such that

$$\|\vec{x} - \vec{a}\| < \delta \Rightarrow \|f(\vec{x}) - f(\vec{a})\| < \varepsilon.$$

Choose $\varepsilon = 1/n$ for $n \in \mathbb{N}$. Then, $\exists$ $\delta(n)$ such that

$$\|\vec{x} - \vec{a}\| < \delta(n) \Rightarrow \|f(\vec{x}) - f(\vec{a})\| < \frac{1}{n}$$

Since $\vec{x}_n \to \vec{a}$, $\exists$ $N_{\delta(n)} \in \mathbb{N}$ such that

$$n \geq N_{\delta(n)} \Rightarrow \|\vec{x}_n - \vec{a}\| < \delta(n)$$

Then for this $N_{\delta(n)}$,

$$n \geq N_{\delta(n)} \Rightarrow \|\vec{x}_n - \vec{a}\| < \delta(n) \Rightarrow \|f(\vec{x}_n) - f(\vec{a})\| < 1/n$$

This implies $f(\vec{x}_n) \to f(\vec{a})$.