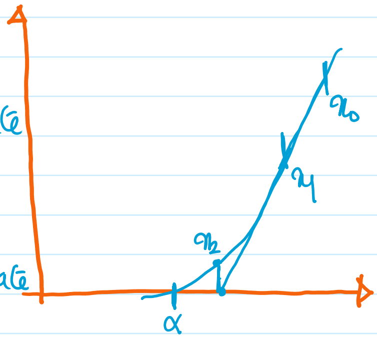
Secant method



Suppose that x_0 and x_1 are the two initial estimates of the root α. We approximate the function by the secant line that connects $(x_0, f(x_0))$ and $(x_1, f(x_1))$. The x-intercept of this line is the next approximate root.

The equation of the secant line is given by

$$y = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

The x-intercept, denoted by x_2, is given by

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

This procedure can be generalised as:

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Note: Recall the Newton's method:

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

$$\approx x_n - \frac{f(x_n)}{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}} = \underbrace{x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}}_{\text{Secant method.}}$$

Error analysis

Suppose that $e_n = x_n - \alpha$. Then

$$e_{n+1} = x_{n+1} - \alpha = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} - \alpha$$

$$= \frac{\cancel{x_n f(x_n)} - x_n f(x_{n-1}) - \cancel{x_n f(x_n)} + x_{n-1} f(x_n)}{f(x_n) - f(x_{n-1})} - \alpha$$

$$= \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{\quad} - \alpha$$

$$= \frac{f(x_n)\,x_{n-1} - f(x_{n-1})\,x_n}{f(x_n) - f(x_{n-1})} - \alpha$$

$$= \frac{f(x_n)\,x_{n-1} - f(\alpha_{n-1})\,x_n - \alpha f(x_n) + \alpha f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$= \frac{f(\alpha_n)\,(x_{n-1} - \alpha) - f(x_{n-1})\,(x_n - \alpha)}{f(x_n) - f(x_{n-1})}$$

$$= \frac{f(x_n)\,e_{n-1} - f(x_{n-1})\,e_n}{f(x_n) - f(x_{n-1})}$$

$$= \frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{f(x_n) - f(x_{n-1})} \times e_n e_{n-1}$$

$$= \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \times \frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}} \times e_n e_{n-1}$$

By Taylor's theorem:

$$f(x_n) = f(\alpha + e_n)$$
$$= f(\alpha) + e_n f'(\alpha) + \frac{1}{2} e_n^2 f''(\alpha) + O(e_n^3)$$
$$= e_n f'(\alpha) + \frac{1}{2} e_n^2 f''(\alpha) + O(e_n^3)$$

$$f(x_n)/e_n = f'(\alpha) + \frac{1}{2} e_n f''(\alpha) + O(e_n^2)$$

Changing the index $n \mapsto n-1$

$$f(x_{n-1})/e_{n-1} = f'(\alpha) + \frac{1}{2} e_{n-1} f''(\alpha) + O(e_{n-1}^2)$$

$$\Rightarrow f(x_n)/e_n - f(x_{n-1})/e_{n-1} = \frac{1}{2} f''(\alpha)\,(e_n - e_{n-1}) + O(e_n^2) - O(e_{n-1}^2)$$
$$\approx \frac{1}{2} f''(\alpha)\,(x_n - x_{n-1})$$

$$\Rightarrow \frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}} \approx \frac{1}{2} f''(\alpha)$$

$$\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \approx \frac{1}{f'(\alpha)}$$

Combine all these estimates to arrive at

$$e_{n+1} \approx \frac{f''(\alpha)}{2f'(\alpha)} e_n e_{n-1} = C e_n e_{n-1}$$

this gives a relationship on how error propagates for the secant method. To get an estimate on the order of convergence, we postulate the following anzats on the asymptotic convergence

$$|e_{n+1}| \sim A|e_n|^{\alpha}$$

$$|e_n| \sim A|e_{n-1}|^{\alpha} \Rightarrow |e_{n-1}| \sim (A^{-1}|e_n|)^{\frac{1}{\alpha}}$$

Thus

$$|e_{n+1}| = A|e_n|^{\alpha} = C|e_n| \times (CA^{-1}|e_n|)^{1/\alpha}$$

$$= CA^{-1/\alpha}|e_n|^{1+1/\alpha}$$

$$\Rightarrow A^{1+1/\alpha}|C|^{-1} = |e_n|^{1-\alpha+1/\alpha}$$

Note that LHS is a constant, so as $n \to \infty$, we should have

$$1-\alpha+1/\alpha = 0 \quad \to \quad \alpha = (1+\sqrt{5})/2 \approx 1.62$$

taking the positive root. Thus the rate of convergence of the secant method is superlinear.

$$A^{1+1/\alpha}|C|^{-1} = 1 \Rightarrow A = |C|^{1/(1+1/\alpha)} = |C|^{1/\alpha}$$

$$= |C|^{\alpha-1}$$

$$= |C|^{0.62}$$

$$= \left|\frac{f''(\alpha)}{2f'(\alpha)}\right|^{0.62}$$

Comparison between Newton's method, secant method, and bisection method.

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Since $(1+\sqrt{5})/2 \approx 1.62 < 2$, secant method is apparently slower than Newton's rate (quadratic), but better than bisection method (linear).

However, each step of secant method only involves one function evaluation; $f(x_n)$; whereas each step of Newton algorithm requires two function evaluations: $f(x_n)$ and $f'(x_n)$. Thus a pair of secant iterations is comparable to one step in the Newton method.

For two steps of the secant method, we have

$$
\begin{aligned}
|e_{n+2}| \sim A|e_{n+1}|^\alpha &= A\,|\,A|e_n|^\alpha|^\alpha \\
&= A^{1+\alpha}|e_n|^{\alpha^2} \\
&= A^{1+\alpha}|e_n|^{(3+\sqrt{5})/2}
\end{aligned}
$$

This is better than the quadratic convergence of the Newton's method.

Theorem. Assume that $f(x)$, $f'(x)$, and $f''(x)$ are continuous for all values of x in some interval containing α, and assume that $f'(\alpha) \neq 0$. Then if the initial guesses x_0 and x_1 are sufficiently close to α, then the iterates x_n will converge to α. The order of convergence will be $p = (1+\sqrt{5})/2 \approx 1.62$.

Proof. Since $f'(\alpha) \neq 0$, there exists $\varepsilon > 0$ such that

$$f'(x) \neq 0 \quad \text{on} \quad I = [\alpha-\varepsilon, \alpha+\varepsilon].$$

Then

$$M := 2\max_{x\in I}|f''(x)| \Big/ \min_{x\in I}|f'(x)|$$

We have for $n \geq 1$

$$e_{n+1} = e_n e_{n-1} \times \left[\frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}}\right] \Bigg/ \left[\frac{f(x_n) - f(x_{n-1})}{\ }\right.$$

$$x_{n+1} = x_n - x_{n-1} \ldots \Big[\frac{\ldots}{x_n - x_{n-1}} \Big] \Big/ \Big[\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \Big]$$

Exc. Show that there exists a point ξ_n between x_n, x_{n-1}, and α such that

$$\frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}} = 2f''(\xi_n)$$

Hint: Use $f(\alpha) = 0$, and Taylor's thm.

By MVT,

$$\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = f'(\xi_n)$$

for some ξ_n between x_n and x_{n-1}

$$\Rightarrow e_{n+1} = e_n e_{n-1} \times \frac{2f''(\xi_n)}{f'(\xi_n)}.$$

This formula implies for x_0 and $x_1 \in I$ and $n = 1$,

$$e_2 = e_1 \cdot e_0 \times 2f''(\xi_1)/f'(\xi_1)$$

$$\Rightarrow |e_2| \leq |e_1| \cdot |e_0| \cdot M$$

$$\Rightarrow M|e_2| \leq M|e_1| \times M|e_0|$$

We choose x_0 and x_1 such that

$$\delta := \max(M|e_0|, M|e_1|) < 1$$

$$\Rightarrow M|e_2| \leq \delta^2 < \delta < 1$$

$$\Rightarrow |e_2| < \delta/M = \max(|e_0|, |e_1|) \leq \varepsilon$$

$$\Rightarrow x_2 \in [\alpha - \varepsilon, \alpha + \varepsilon]$$

This argument applied inductively, shows that

$$x_n \in [\alpha - \varepsilon, \alpha + \varepsilon] \text{ and } M|e_n| \leq \delta^2 < \delta \quad \forall\, n \geq 2.$$

Now note that

$$M|e_3| \leq M|e_2| \cdot M|e_1| \leq \delta^2 \cdot \delta = \delta^3$$
$$M|e_4| \leq M|e_3| \cdot M|e_2| \leq \delta^3 \cdot \delta^2 = \delta^5$$

If $M|e_n| \leq \delta^{q_n}$, then

$$M|e_{n+1}| \leq M|e_n| \cdot M|e_{n-1}| \leq \delta^{q_n} \cdot \delta^{q_{n-1}} = \delta^{q_n + q_{n-1}} = \delta^{q_{n+1}}$$

If $M|e_n| \le \delta^{q_n}$, then

$$M|e_{n+1}| \le M|e_n| \cdot M|e_{n-1}| \le \delta^{q_n} \cdot \delta^{q_{n-1}} = \delta^{q_n + q_{n-1}} = \delta^{q_{n+1}}$$

$$\Rightarrow \quad q_{n+1} = q_n + q_{n-1} \qquad n \ge 1$$

$$q_0 = q_1 = 1$$

This is a Fibonacci sequence of numbers and

$$q_n = \frac{1}{\sqrt{5}} \left[r_0^{n+1} - r_1^{n+1} \right] \qquad n \ge 0$$

with $r_0 = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $r_1 = \frac{1-\sqrt{5}}{2} \approx -0.618$

Therefore, for large n, $\quad q_n \sim \frac{1}{\sqrt{5}} r_0^{n+1}$.

Now, $\quad |e_n| \le \frac{1}{M} \delta^{q_n}$. Since $\delta < 1$ and $q_n \to \infty$ as $n \to \infty$,

$|e_n| \to 0$ as $n \to \infty$, this means that $a_n \to \alpha$ as $n \to \infty$.

Now we show that the bounds satisfy the order of convergence

$p = r_0 = \frac{1+\sqrt{5}}{2}$.

$$\frac{B_{n+1}}{B_n^{r_0}} = \frac{1/M \ \delta^{q_{n+1}}}{(1/M)^{r_0} \ \delta^{r_0 q_n}} = M^{r_0 - 1} \delta^{q_{n+1} - r_0 q_n}$$

$$\le \delta^{-1} M^{r_0 - 1} = c$$

[verify: Hint: $q_{n+1} - r_0 q_n = r_1^{n+1} > -1$]

$$\Rightarrow \quad B_{n+1} \le c B_n^{r_0}$$

$\Rightarrow$ the order of convergence of the bounds is $p = r_0 = (1+\sqrt{5})/2$.

In fact, we can show that, $\quad \lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^{r_0}} = \left| \frac{f''(\alpha)}{2 f'(\alpha)} \right|^{(\sqrt{5}-1)/2}$

This proof is omitted for this course.

$\square$