Aitken Extrapolation and Multiple Roots

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Aitken extrapolation for linearly convergent sequences.

Recall from the theory fixed point methods, if α is a fixed point of

$$g : [a,b] \to [a,b]$$

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha)$$

for the convergent iteration $x_{n+1} = g(x_n)$.

In this section, we restrict our focus to linear convergence and assume that $0 < |g'(\alpha)| < 1$. Our aim is to device a way to accelerate the convergence.

Consider the ratio

$$\lambda_n := \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}} \qquad \text{with } n \geq 2$$

We claim that $\lim_{n\to\infty} \lambda_n = g'(\alpha)$. To see this, write

$$\lambda_n = \frac{(\alpha - x_{n-1}) - (\alpha - x_n)}{(\alpha - x_{n-2}) - (\alpha - x_{n-1})}$$

$$= \frac{(\alpha - x_{n-1}) - (g(\alpha) - g(x_{n-1}))}{(\alpha - x_{n-2}) - (\alpha - x_{n-1})}$$

$$= \frac{(\alpha - x_{n-1}) - g'(\xi_{n-1})(\alpha - x_{n-1})}{(\alpha - x_{n-2}) - (\alpha - x_{n-1})}$$

Now $\alpha - x_{n-1} = g(\alpha) - g(x_{n-2}) = g'(\xi_{n-2})(\alpha - x_{n-2})$

$$\Rightarrow \quad \lambda_n = \frac{(\alpha - x_{n-1}) - g'(\xi_{n-1})(\alpha - x_{n-1})}{(\alpha - x_{n-1})/g'(\xi_{n-2}) - (\alpha - x_{n-1})}$$

$$\Rightarrow \quad \lambda_n = \frac{1 - g'(\xi_{n-1})}{1/g'(\xi_{n-2}) - 1}$$

$$\frac{\cdots}{1/g'(\xi_{n-2}) - 1}$$

$$\Rightarrow \quad \lim_{n\to\infty} \lambda_n = \frac{1 - g'(\alpha)}{1/g'(\alpha) - 1} = g'(\alpha)$$

Now note that $\quad \alpha - x_n = g(\alpha) - g(x_{n-1})$

$$\approx g'(\alpha)\,(\alpha - x_{n-1})$$

$$\approx \lambda_n\,(\alpha - x_{n-1})$$

$$\Rightarrow \quad \alpha - x_n = (\alpha - x_{n-1}) + (x_{n-1} - x_n)$$

$$= \frac{1}{\lambda_n}(\alpha - x_n) + (x_{n-1} - x_n)$$

Re-arrange the terms to arrive at

$$\alpha - x_n \approx \frac{\lambda_n}{1-\lambda_n}\,(x_n - x_{n-1})$$

This is called Aitken's error formula and it becomes increasingly accurate as $n \to \infty$ and $\lambda_n \to g'(\alpha)$.

$$\alpha \approx x_n + \frac{\lambda_n}{1-\lambda_n}(x_n - x_{n-1}) \qquad \text{for } n \geq 2$$

$$= x_n - \frac{(x_n - x_{n-1})^2}{(x_n - x_{n-1}) - (x_{n-1} - x_{n-2})} \qquad \text{for } n \geq 2$$

$$=: \hat{x}_n$$

This is called Aitken's extrapolation formula, which gives a better estimate of α.

The Aitken's extrapolation algorithm is as follows:

Step 1: $x_1 = g(x_0)$, $x_2 = g(x_1)$

Step 2: $\hat{x}_2 = x_2 - \dfrac{(x_2 - x_1)^2}{(x_2 - x_1) - (x_1 - x_0)}$

Step 3: If $|\hat{x}_2 - x_2| \leq \varepsilon$, then root $= \hat{x}_2$ and exit.

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Step 4: Set $x_0 = \hat{x}_2$ and go to step 1.

Multiple roots and Newton's method

Suppose that a function f has a root α of multiplicity p. This means the function f has the form

$$f(x) = (x-\alpha)^p h(x)$$

with $h(\alpha) \neq 0$ and $h(x)$ is continuous at $x = \alpha$.

The root finding problem $f(x) = 0$ can be reformulated as a fixed point problem

$$x_{n+1} = g(x_n) \quad \text{with} \quad g(x) = x - \frac{f(x)}{f'(x)}.$$

Then

$$g(x) = x - \frac{(x-\alpha)\, h(x)}{p h(x) + (x-\alpha) h'(x)}$$

(Verify by computing $f'(x)$)

$$g'(x) = 1 - \frac{h(x)}{p h(x) + (x-\alpha) h'(x)} - (x-\alpha) \frac{d}{dx}\left(\frac{h(x)}{p h(x) + (x-\alpha) h'(x)} \right)$$

$\Rightarrow$ $g'(\alpha) = 1 - 1/p$. Thus Newton's method in this case is a linear scheme with a rate of convergence $1 - 1/p$.

We can improve the convergence by changing g. Define g as

$$\boxed{g(x) = x - p\, f(x)/f'(x)}.$$

This implies $g'(\alpha) = 0$ (verify). Thus

$$\alpha - x_{n+1} = g(\alpha) - g(x_n)$$

$$= -g'(\alpha)(x_n - \alpha) - \frac{1}{2}(x_n - \alpha)^2 g''(\xi_n)$$

with ξ_n b/w x_n & α.

$$\Rightarrow \quad \alpha - x_{n+1} = -\frac{1}{2}(\alpha - x_n)^2 g''(\xi_n)$$

$\Rightarrow$ Convergence is quadratic.