Interpolation

Given $(n+1)$ data points $\{(x_i, y_i)\}_{0 \le i \le n}$, interpolation seeks ~~a~~ the polynomial of least degree such that

$$p(x_i) = y_i \qquad 0 \le i \le n.$$

We say that the polynomial interpolate the given data.

Theorem. If $x_0, x_1, \dots, x_n$ are distinct real numbers, then for arbitrary values $y_0, y_1, \dots, y_n$, there exists a unique polynomial p_n of degree at most n such that

$$p_n(x_i) = y_i \qquad 0 \le i \le n$$

Proof. Uniqueness. Suppose that there exists two polynomials p_n and q_n of degree at most n such that $p_n(x_i) = y_i = q_n(x_i)$. Then

$$r_n(x_i) = p_n(x_i) - q_n(x_i) = 0 \quad \forall\, 0 \le i \le n.$$

Thus r_n ~~a~~ is an polynomial of degree at the most n with ~~zeros~~ $n+1$ zeros. This is only possible if $r_n(x) \equiv 0$. Therefore $p_n(x) \equiv q_n(x)$.

Existence: This follows from induction. For the base case consider $n=0$. Then consider the constant function

$$p_0(x) = y_0$$

Note that p_0 is of degree ≤ 0 and $p_0(x_0) = y_0$.
For the inductive case, assume that p_{k-1} is a polynomial of degree $\leq k-1$ with $p_{k-1}(x_i) = y_i$ for $0 \leq i \leq k-1$.

Define the function

$$p_k(x) = p_{k-1}(x) + \frac{y_k - p_{k-1}(x_k)}{(x_k - x_0)\cdots(x_k - x_{k-1})} \times (x - x_0)\cdots(x - x_{k-1})$$

Then $p_k(x)$ is a polynomial of degree $\leq k$. Furthermore p_k interpolates the data points of p_{k-1} because

$$p_k(x_i) = p_{k-1}(x_i) = y_i \quad \forall\ 0 \leq i \leq k-1$$

Moreover

$$\begin{aligned} p_k(x_k) &= p_{k-1}(x_k) + y_k - p_{k-1}(x_k) \\ &= y_k. \end{aligned}$$

Thus p_k interpolates the data point (x_k, y_k) also. ∎

<u>Newton's form of interpolating polynomial</u>

The constructive proof mentioned above gives a polynomial p_k which is precisely equal to p_{k-1} plus an additional term $c_k(x - x_0)\cdots(x - x_k)$ with

$$c_k = \frac{y_k - p_{k-1}(x_k)}{(x_k - x_0)\cdots(x_k - x_{k-1})}$$

In fact each $p_k(x)$ has the form

$$p_k(x) = c_0 + c_1(x - x_0) + \cdots + c_k(x - x_0)\cdots(x - x_{k-1})$$

The compact form of this is

$$p_k(x) = \sum_{i=0}^{k} c_i \prod_{j=0}^{i-1} (x - x_j)$$

with the convention $\prod_{j=0}^{m} (x-x_j) = 1$ whenever $m < 0$.

The first few cases are

$$p_0(x) = c_0$$
$$p_1(x) = c_0 + c_1(x - x_0)$$
$$p_2(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1)$$

These polynomials are called interpolation polynomials in Newton's form.

<u>Lagrange form</u>

Lagrange form seeks a polynomial of the form:

$$p(x) = y_0 \ell_0(x) + y_1 \ell_1(x) + \cdots + y_n \ell_n(x)$$
$$= \sum_{k=0}^{n} y_k \ell_k(x)$$

Such that ~~ℓ_k~~ $\ell_k(x)$ is zero for $x_i \neq x_k$ and $\ell_k(x_k) = 1$. For instance consider the case ℓ_0. Note that ℓ_0 is an nth degree polynomial vanishing at $x_i \neq x_0$ ~~and~~ for $1 \leq i \leq n$ and $\ell_0(x_0) = 1$. Thus ℓ_0 has the form

$$\ell_0(x) = c(x-x_1) \cdots (x - x_n)$$

$$= c \prod_{j=1}^{n} (x - x_i)$$

$$l_0(x_0) = c \prod_{i=1}^{n} (x_0 - x_i) = 1$$

$$\Rightarrow c = 1 / \prod_{i=1}^{n} (x_0 - x_i)$$

Therefore,

$$l_0(x_0) = \prod_{i=1}^{n} \frac{x - x_i}{x_0 - x_i} .$$

In general

$$l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

~~These~~ For a set of nodes $x_0, x_1, \cdots, x_n$ these polynomials are known as cardinal functions. This form of polynomial is called Lagrange polynomial.

Theorem (polynomial interpolation error). Let f be a function in $C^{n+1}[a,b]$, and let p be a the polynomial of degree at most n that interpolates the function f at $n+1$ distinct points $x_0, x_1, \cdots, x_n$ in the interval $[a,b]$. To each x in $[a,b]$, there corresponds a point ξ_x in (a,b) such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

Proof. If x is one of the nodes of interpolation x_i, then both LHS and RHS are equal to zero. Suppose that x be any ~~other~~ point other than a node. Put

$$w(t) = \prod_{i=0}^{n} (t - x_i)$$

Let $\phi = f - p - \lambda w$, where λ is a ~~poin~~ real number that makes $\phi(x) = 0$. Thus

$$\lambda = \frac{f(x) - p(x)}{w(x)}$$

Note that ϕ vanishes at $n+2$ points $x, x_0, \dots, x_n$. By Rolles theorem ϕ' has at least $n+1$ distinct zeros in (a,b). Similarly ϕ'' has at least n distinct zeros in (a,b). By repeated argument, we conclude that $\phi^{(n+1)}$ has at least one zero, say ξ_x, in (a,b). Now

$$\phi^{(n+1)} = f^{(n+1)} - p^{(n+1)} - \lambda w^{(n+1)}$$

$$= f^{(n+1)} - (n+1)!\,\lambda \qquad \text{(verify)}.$$

Hence $$0 = \phi^{(n+1)}(\xi_x) = f^{(n+1)}(\xi_x) - (n+1)!\,\lambda.$$

$$= f^{(n+1)}(\xi_x) - (n+1)!\,\frac{f(x) - p(x)}{w(x)}.$$

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