Polynomial Interpolation Error

11 March 2026 11:03

Theorem. (Polynomial interpolation error). Let f be a function in $C^{n+1}[a,b]$, and let p be a polynomial of degree at most n that interpolates the function f at $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in the interval $[a,b]$. To each x in $[a,b]$, there corresponds a point ξ_x in (a,b) such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

Proof. If x is one of the nodes of interpolation x_i, then both LHS and RHS are equal to zero. Suppose that x be any point other than a node. Define

$$w(t) = \prod_{i=0}^{n} (t - x_i)$$

and $\phi = f - p - \lambda w$, where λ is a real number that makes $\phi(x) = 0$. Thus

$$\lambda = \frac{f(x) - p(x)}{w(x)}$$

Note that ϕ vanishes at $n+2$ points $x, x_0, x_1, \ldots, x_n$. By Rolle's thm, ϕ' has atleast $n+1$ distinct zeros in (a,b). By repeated argument, conclude that $\phi^{(n+1)}$ has atleast one zero, say ξ_x, in (a,b).

Now

$$\phi^{(n+1)} = f^{(n+1)} - p^{(n+1)} - \lambda w^{(n+1)}$$

$$= f^{(n+1)} - (n+1)!\,\lambda \qquad \text{(Verify)}$$

$$\Rightarrow \phi^{(n+1)}(\xi_x) = f^{(n+1)}(\xi_x) - (n+1)!\,\lambda = 0$$

$$\Rightarrow f^{(n+1)}(\xi_x) - (n+1)! \frac{f(x) - p(x)}{w(x)} = 0$$

$$\Rightarrow f(x) - p(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

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