Chebychev Polynomials

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In the polynomial interpolation error

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

the product term can be optimized by choosing the nodes in a special way. We shall prove that if the nodes x_i are the roots of the Chebychev polynomial, then the error will be the least.

The Chebychev polynomials are defined by the recurrence formula

$$\boxed{\begin{aligned} &T_0(x) = 1, \quad T_1(x) = x \\ &T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x) \end{aligned}}$$

$$\begin{aligned} T_2(x) &= 2x T_1(x) - T_0(x) \\ &= 2x \cdot x - 1 = 2x^2 - 1 \end{aligned}$$

$$\left.\begin{aligned} T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \end{aligned}\right\} \text{verify.}$$

Theorem (Closed form of Chebychev polynomials) For x in the interval $[-1, 1]$, the Chebybychev polynomials have the closed form expression

$$T_n(x) = \cos(n \cos^{-1} x) \qquad (n \geq 0)$$

Proof. Recall the addition formula for the cosine:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

this shows that

$$\cos(n+1)\theta = \cos\theta \cos n\theta - \sin\theta \sin n\theta$$

$$\cos(n-1)\theta = \cos\theta \cos n\theta + \sin\theta \sin n\theta$$

Add these two expressions to arrive at

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$$
$$\Rightarrow \cos(n+1)\theta = 2\cos\theta\cos n\theta - \cos(n-1)\theta$$

Let $\theta = \cos^{-1}x$ for $x \in [-1, 1]$ $(x = \cos\theta)$ and $f_n(x) = \cos(n\cos^{-1}x)$

$$f_0(x) = \cos(0 \times \cos^{-1}x) = 1$$
$$f_1(x) = \cos(\cos^{-1}x) = x$$

$$f_{n+1}(x) = \cos((n+1)\cos^{-1}x) = \cos((n+1)\theta)$$
$$= 2\cos\theta\cos n\theta - \cos(n-1)\theta$$
$$= 2x\cos(n\cos^{-1}x) - \cos((n-1)\cos^{-1}x)$$
$$= 2x f_n(x) - f_{n-1}(x)$$

Therefore $f_n = T_n$ for all n. That is $T_n(x) = \cos(n\cos^{-1}x)$.

The last proved Theorem shows that, the Chebychev polynomials satisfy

$$|T_n(x)| \leq 1 \qquad (-1 \leq x \leq 1)$$
$$T_n\left(\cos\frac{j\pi}{n}\right) = (-1)^j \qquad (0 \leq j \leq n)$$
$$T_n\left(\cos\frac{2j-1}{2n}\pi\right) = 0 \qquad (1 \leq j \leq n)$$

Note that the n^{th} Chebychev polynomial $T_n(x)$ has the leading order term $2^{n-1}x^n$ for $n > 0$. Therefor $2^{1-n}T_n$ is a monic polynomial.

Theorem. If p is a monic polynomial of degree n, then

$$\|p\|_\infty = \max_{-1 \leq x \leq 1} |p(x)| \geq 2^{1-n}.$$

Proof. Proof is through contradiction. Suppose that

$$\|p\|_\infty < 2^{1-n}. \quad \text{for } |x| \leq 1.$$

$$q(x_i) = 2^{1-n}T_n(x_i)$$
$$= 2^{1-n}(-1)^i$$

Let $q = 2^{1-n}T_n$ and $x_i = \cos\left(\frac{i\pi}{n}\right)$.

$$(-1)^i p(x_i) \leq |p(x_i)| < 2^{1-n} = (-1)^i q(x_i)$$

$$\Rightarrow \quad (-1)^i \left[p(x_i) - q(x_i) \right] < 0$$
$$(-1)^i \left[q(x_i) - p(x_i) \right] > 0 \qquad (0 \le i \le n)$$

This shows that the polynomial $q-p$ flips its sign $n+1$ times in the interval $[-1,1]$. Therefore it must have at least n roots in the interval $(-1,1)$. But the degree of $q-p$ is at the most $n-1$. Therefore $p-q$ can have at the most $n-1$ roots. This is a contradiction.

Selection of nodes.

Assume that the interpolation nodes are in the interval $[-1,1]$. If x is also in the same interval, ξ_x will also be in the same interval. Therefore,

$$\max_{-1 \le x \le 1} |f(x) - p(x)| \le \frac{1}{(n+1)!} \max_{|x| \le 1} |f^{n+1}(x)| \times \max_{|x|\le 1} \left| \prod_{i=0}^{n} (x - x_i) \right|$$

By the previous theorem

$$\max_{|x| \le 1} \left| \prod_{i=0}^{n} (x - x_i) \right| \ge 2^{-n}$$

For the Chebychev polynomial of degree $n+1$

$$\| 2^{-n} T_{n+1} \|_\infty \ge 2^{-n}$$

$$T_{n+1}\left(\cos j\pi/n\right) = (-1)^j \quad \text{for} \quad 0 \le j \le n$$

$$\Rightarrow \quad \| T_{n+1} \|_\infty = 1$$

$$\Rightarrow \quad \| 2^{-n} T_{n+1} \|_\infty = 2^{-n}$$

Therefore the minimum is attained for the Chebychev polynomials.

$$\prod_{i=0}^{n} (x - x_i) = 2^{-n} T_{n+1}(x)$$

Therefore the nodes x_i are the roots of the polynomial T_{n+1}; which are

$$x_i = \cos\left(\frac{2i+1}{2n+2}\pi\right) \qquad 0 \le i \le n.$$

Theorem. If the nodes x_i are the roots of the Chebyshev polynomial T_{n+1}, then the error formula in Thm (polynomial error) yields for $|x| < 1$

$$|f(x) - p(x)| \le \frac{1}{2^n (n+1)!} \max_{|t|<1} |f^{n+1}(t)|$$

Convergence of interpolating polynomial

If a continuous function f is prescribed on an interval $[a,b]$, and if the interpolating polynomials p_n of higher degrees are constructed, we expect that $\|f - p_n\|_\infty$ goes to zero. Then is not true in general.

Runge Phenomenon

Consider the function $f(x) = (x^2+1)^{-1}$ on $[-5,5]$. Suppose that interpolating polynomials p_n are constructed using equally spaced nodes in $[-5,5]$. We can see that $\|f - p_n\|_\infty$ is unbounded. This is called Runge Phenomenon.

Theorem (Fabers Thm) For any prescribed system of nodes

$$a \le x_0^{(n)} < x_1^{(n)} < \cdots < x_n^{(n)} \le b \qquad (n \ge 0),$$

there exists a continuous function f on $[a,b]$ such that the interpolating polynomials for f using these nodes fail to converge uniformly to f.