Weierstrass Approximation Theorem

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Faber's theorem states that for any system of interpolating nodes, there exists a continuous function whose interpolating polynomial fails to converge. This is a negative result. We also have a positive result.

Theorem. (Convergent polynomials) If f is a continuous function on $[a,b]$, there exists a system of nodes such that the polynomials p_n of the interpolation to f at these nodes satisfy

$$\lim_{n\to\infty} \| f - p_n \|_\infty = 0.$$

Theorem (Weierstrass Approximation Theorem). If f is continuous on $[a,b]$ and if $\varepsilon > 0$, then there exists a polynomial p satisfying

$$|f(x) - p(x)| \leq \varepsilon$$

on the interval $[a,b]$.

Proof. We shall prove the Theorem on $[0,1]$. For a proof on $[a,b]$, we can use the change of variable $x = a + t(b-a)$ and the observation that polynomial of a linear function of t is a polynomial in t of the same degree.

If $f \in C[0,1]$, the sequence of Bernstein polynomials converge uniformly to f. The Bernstein polynomials are defined by

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) g_{nk}(x)$$

$$g_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & 0 \leq k \leq n \\ 0 & \text{ow.} \end{cases}$$

Note that $B_n : C[0,1] \longrightarrow C[0,1]$ is a linear operator. That is

$$B_n(af + bg) = aB_n f + bB_n g \qquad (a,b \in \mathbb{R},\ f,g \in C[0,1])$$

Also note that B_n is a positive operator. That is $f \geq 0$, then $B_n f \geq 0$.

Theorem (Bohman–Korovkin Thm). Let L_n $(n \geq 1)$ be a sequence of positive linear operators defined on $C[a,b]$ and taking values in the same space. If $\|L_n f - f\|_\infty \to 0$ for the three functions

$f(x) = 1, x, x^2$, then the same is true for all $f \in C[a,b]$.

Proof. Consider the function $|f|$. Suppose L is positive linear op^{tr}. If $f \geq g$, then $Lf \geq Lg$.

Since $|f| \geq f$, it follows $L(|f|) \geq L(f)$
$L(|f|) \geq L(-f)$

Set $h_k(x) = x^k$ for $k = 0, 1, 2$. Define the functions α_n, β_n, and γ_n by

$$\alpha_n := L_n h_0 - h_0 \qquad \beta_n := L_n h_1 - h_1 \qquad \gamma_n := L_n h_2 - h_2$$

From the hypothesis:

$$\|\alpha_n\|_\infty \to 0 \qquad \|\beta_n\|_\infty \to 0 \qquad \|\gamma_n\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Assume that f is an arbitrary element of $C[a,b]$. Let $\varepsilon > 0$. Our intention is to prove that, $\exists\, m \in \mathbb{N}$ such that

$$n \geq m \implies \|L_n f - f\|_\infty < \varepsilon$$

Since f is continuous on $[a,b]$, it is uniformly continuous. Therefore for a given $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Set $c = 2\|f\|_\infty / \delta^2$. Then for $|x - y| \geq \delta$, it follows:

$$|f(x) - f(y)| \leq 2\|f\|_\infty \leq 2\|f\|_\infty \frac{(x-y)^2}{\delta^2} = c(x-y)^2$$

Therefore, for all $x, y \in [a,b]$, we have

$$|f(x) - f(y)| \leq \varepsilon + c(x-y)^2$$

Since $h_0(x) = 1$, the last inequality can be written as

$$|f(x) - f(y) h_0(x)| \leq \varepsilon h_0(x) + c[h_2(x) - 2y h_1(x) + y^2 h_0(x)]$$

$$\Rightarrow |L_n f(x) - f(y) L_n h_0(x)| \leq \varepsilon L_n h_0(x) + c[L_n h_2(x) - 2y L_n h_1(x) + y^2 L_n h_0(x)]$$

use $x = y$ to arrive at

$$|L_n f(y) - f(y) L_n h_0(y)| \leq \varepsilon L_n h_0(y) + c[L_n h_2(y) - 2y L_n h_1(y) + y^2 L_n h_0(y)]$$

$$= \varepsilon [L_n h_0 - h_0 + h_0](y) + c\Big[(L_n h_2 - h_2 + h_2)(y) - 2y [(L_n h_1 - h_1 + h_1)(y)] + y^2 [(L_n h_0 - h_0 + h_0)(y)]\Big]$$

$$= \varepsilon [\alpha_n(y) + 1] + c\Big[(\gamma_n(y) + y^2) - 2y(\beta_n(y) + y) + y^2(\alpha_n(y) + 1)\Big]$$

$$= \varepsilon + \varepsilon \alpha_n(y) + c\gamma_n(y) - 2cy \beta_n(y) + cy^2 \alpha_n(y)$$

$$< \varepsilon + \varepsilon \|\alpha_n\|_\infty + c\|\gamma_n\|_\infty + 2c\|h_1\|_\infty \|\beta_n\|_\infty + c\|h_2\|_\infty \|\alpha_n\|_\infty$$

Since $\|h_1\|_\infty$ and $\|h_2\|_\infty$ are bounded and $\|\alpha_n\|_\infty, \|\beta_n\|_\infty, \|\gamma_n\|_\infty$ go to zero as $n \to \infty$, we can choose $m \in \mathbb{N}$ such that $m \geq n \Rightarrow$ the RHS of the last inequality $\leq 3\varepsilon$.

$$\Rightarrow \|L_n f - f L_n h_0\| \leq 3\varepsilon$$

Therefore,

$$\begin{aligned} \|L_n f - f\|_\infty &\leq \|L_n f - f L_n h_0\|_\infty + \|f L_n h_0 - f h_0\|_\infty \\ &\leq 3\varepsilon + \|f\|_\infty \|L_n h_0 - h_0\|_\infty \\ &\leq 3\varepsilon + \varepsilon = 4\varepsilon \to 0 \end{aligned}$$

$\boxtimes$