Therefore, for $n \geq m$ we have

$$\| L_n f - f L_n h_0 \|_\infty \leq 3\varepsilon$$

Therefore

$$\| L_n f - f \|_\infty \leq \| L_n f - f \cdot L_n h_0 \|_\infty + \| f L_n h_0 - f h_0 \|_\infty$$

$$\leq 3\varepsilon + \| f \|_\infty \| L_n h_0 - h_0 \|_\infty$$

$$\leq 3\varepsilon + \varepsilon = 4\varepsilon.$$

☑

Thus for Bernstein polynomials it is enough to show that $B_n h_0 \to h_0$, $B_n h_1 \to h_1$ and $B_n h_2 \to h_2$.

$$(B_n h_0)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x + (1-x))^n = 1$$

$$(B_n h_1)(x) = \sum_{k=0}^{n} \left[\frac{k}{n}\right] \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k}$$

$$= x \sum_{k=0}^{n-1} \binom{n-1}{k-1} x^k (1-x)^{n-1-k} = x.$$

Verify:

$$(B_n h_2)(x) = \sum_{k=0}^{n} \left[\frac{k}{n}\right]^2 \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{k=1}^{n} \left[\frac{k}{n}\right] \binom{n-1}{k-1} x^k (1-x)^{n-k}$$

$$= \sum_{k=1}^{n} \left[\frac{n-1}{n} \frac{k-1}{n-1} + \frac{1}{n}\right] \binom{n-1}{k-1} x^k (1-x)^{n-k}$$

$$= \frac{n-1}{n} x^2 \sum_{k=2}^{n} \binom{n-2}{k-2} x^{k-2} (1-x)^{n-k} + x/n$$

$$= \frac{n-1}{n} x^2 + \frac{x}{n} \to x^2.$$ (Give this an exe. to students).

Divided differences.

Consider a set of $n+1$ distinct nodes $x_0, x_1, \dots, x_n$ and the problem of interpolating a function f on these nodes. We already know that there exists a unique polynomial p of degree at the most n that interpolates f at the $n+1$ nodes:

$$p(x_i) = f(x_i). \qquad (0 \le i \le n)$$

We consider the Newton's form of interpolating polynomial:

$$p(x) = \sum_{j=0}^{n} c_j q_j(x),$$

wherein $q_j(x) = (x-x_0)(x-x_1)\cdots(x-x_{j-1}) = \prod_{k=0}^{j-1}(x-x_k)$

Substituting ~~for each $x = x_i$, we obtain~~

~~$c_0 (x$ $c_0 q_0(x_0) = c_0$~~ The interpolation conditions imply

$$p(x_0) = \sum_{j=0}^{n} c_j q_j(x_0) = f(x_0)$$

$$p(x_1) = \sum_{j=0}^{n} c_j q_j(x_1) = f(x_1)$$

$$\vdots$$

$$p(x_n) = \sum_{j=0}^{n} c_j q_j(x_n) = f(x_n)$$

This leads to an $(n+1) \times (n+1)$ linear system in $c_0, c_1, \ldots, c_n$ with

$$A\vec{c} = \vec{f}$$

wherein $A(i,j) = a_{ij} = q_j(x_i)$ if $0 \le i, j \le n$

This matrix is lower triangular. ∴ if $j > i$, then

$$a_{ij} = \prod_{k=1} q_j(x_i) = \prod_{k=1}^{j-1} (x_i - x_k)$$

Since $i < j$, $\prod_{k=1}^{j-1}(x_i - x_j)$ contains the factor $(x_i - x_i)$

hence is equal to zero. As a concrete example consider the case $n=2$. The ~~linear~~ ~~equal~~ interpolating polynomial is

$$p_2(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1)$$

$$p_2(x_0) = c_0 = f(x_0)$$

$$p_2(x_1) = c_0 + c_1(x_1 - x_0) = f(x_1)$$

$$p_2(x_2) = c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1) = f(x_2).$$

The corresponding lower triangular system is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}$$

Note that solving from top to bottom

$$c_0 = f(x_0), \quad c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and $c_2 = \dfrac{f(x_2) - f(x_0) - (x_2 - x_0)\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{(x_2 - x_0)(x_1 - x_0)}$

Note that c_0 depends on $f(x_0)$, c_1 on $f(x_0)$ and $f(x_1)$, c_2 on $f(x_0)$, $f(x_1)$, and $f(x_2)$. Thus we denote

$$c_n = f[x_0, x_1, \cdots, x_n],$$

which is called the divided difference of f at at $x_0, x_1, \cdots, x_n$. Note that the interpolating polynomial is

$$p(x) = \sum_{j=0}^{n} f[x_0, \cdots x_j]\, q_j(x)$$

Since $q_j(x) = (x - x_0) \cdots (x - x_{j-1})$
$= x^j +$ lower order terms,

$f[x_0, \cdots x_j]$ is the coefficient of x^j in The interpolating polynomial.

Note that $f[x_0] = f(x_0)$. The quantity $f[x_0, x_1]$ is the coefficient of x in the polynomial of degree at most 1 interpolating f at x_0 and x_1. Since that polynomial is

$$p(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_1}(x - x_0)$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Here we introduce the notion of divided difference table

$$\begin{array}{lll} x_0 & f(x_0) & \dfrac{(f(x_0)-f(x_1))}{x_0-x_1} = f[x_0,x_1] \\ x_1 & f(x_1) & \end{array}$$

Then the interpolating polynomial is given by

$$p(x) = f(x_0) + f[x_0,x_1](x-x_0).$$

Now we need the following thm.

Thm. Divided differences satisfy the equation

$$f[x_0,x_1,\ldots,x_n] = \frac{f[x_1,\ldots,x_n] - f[x_0,\ldots,x_{n-1}]}{x_n - x_0}$$

Proof. Let p_k denote the polynomial of degree at most k that interpolates f at the nodes $x_0, x_1, \ldots, x_k$. Let q be the polynomial of degree at most $n-1$ that interpolates f at $x_1, x_2, \ldots, x_n$. Then

$$p_n(x) = q(x) + \frac{(x-x_n)}{x_n-x_0}[q(x) - p_{n-1}(x)] \quad \text{(verify)}$$

The coefficient of x^n on the RHS is given by

$$\frac{\text{coefficient of } x^{n-1} \text{ in } q(x) - \text{coefficient of } x^{n-1} \text{ in } p_{n-1}(x)}{x_n - x_0}$$

$$\Rightarrow f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}$$

This theorem allows for the formula

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$\vdots$$

$$f[x_i, x_{i+1}, \ldots, x_{i+j}] = \frac{f[x_{i+1}, \ldots, x_{i+j}] - f[x_i, \ldots, x_{i+j-1}]}{x_{i+j} - x_i}$$

We can present this as a table called the divided difference table:

x_0	$f[x_0]$			
	~~$f[x_0]$~~	$f[x_0, x_1]$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

The polynomial is obtained from reading the top row

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

Example:

x	3	1	5	6
$f(x)$	1	-3	2	4

x	$f(x)$			
3	1			
		$\frac{-3-1}{1-3} = 2$		
1	-3		$\frac{5/4-2}{5-3} = -3/8$	
		$\frac{2+3}{5-1} = 5/4$		$\frac{3/20+3/8}{6-3} = 7/40$
5	2		$\frac{2-5/4}{6-1} = 3/20$	
		$\frac{4-2}{6-5} = 2$		
6	4			

The interpolating polynomial is given by

$$p(x) = 1 + 2(x-3) - \frac{3}{8}(x-3)(x-1) + \frac{7}{40}(x-3)(x-1)(x-5)$$

Theorem (Permutations). The divided difference is a symmetric function of its arguments. Thus, if $(z_0, z_1, \dots z_n)$ is a permutation of $(x_0, \dots, x_n)$, then

$$f[z_0, \dots, z_n] = f[x_0, \dots, x_n]$$

Proof. Note that $f[z_0, \dots, z_n]$ is the coefficient of x^n in the interpolating polynomial subordinate to the nodes $z_0, \dots z_n$. $f[x_0, \dots, x_n]$ is the coefficient of the x^n in the interpolating polynomial over the nodes $[x_0, \dots, x_n]$. Since these polynomials are one and the same, it follows that RHS = LHS.

Theorem. Let p be the polynomial of degree at most n that interpolates a function f at a set of $n+1$ distinct nodes $x_0, x_1, \dots, x_n$. If t is a point