Properties of divided differences, Hermite-Genocchi Theorem

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Theorem (Permutations). The divided differences is a symmetric function of its arguments. That is if $(z_0, z_1, z_2, \dots, z_n)$ is a permutation of $(x_0, \dots, x_n)$, then

$$f[x_0, x_1, \dots, x_n] = f[z_0, z_1, \dots, z_n] \quad (*).$$

Proof. Note that $f[z_0, z_1, \dots, z_n]$ is the coefficient of x^n in the interpolating polynomial subordinate to the nodes $z_0, z_1, \dots z_n$. Likewise, $f[x_0, \dots x_n]$ is the coefficient of x^n in the interpolating polynomial subordinate to $x_0, \dots, x_n$. Since these polynomials are one and the same, it follows that LHS = RHS in $(*)$.

Theorem. Let p be the polynomial of degree at most n that interpolates a function f at a set of $n+1$ distinct nodes $x_0, \dots, x_n$. If t is a point different from these nodes, then

$$f(t) - p(t) = f[x_0, \dots, x_n, t] \prod_{j=0}^{n} (t - x_j)$$

Proof. First, let q be the polynomial of degree at the most $n+1$ that interpolates f at the nodes $x_0, x_1, \dots, x_n, t$. Therefore

$$q(x) = p(x) + f[x_0, x_1, \dots, x_n, t] \prod_{j=0}^{n} (x - x_j)$$

we have $q(t) = f(t)$. This implies

$$f(t) = q(t) = p(t) + f[x_0, x_1, \dots, x_n, t] \prod_{j=0}^{n} (t - x_j)$$

$$\Rightarrow f(t) - p(t) = f[x_0, x_1, \dots, x_n, t] \prod_{j=0}^{n} (t - x_j).$$

Theorem. If f is n times continuously differentiable on $[a,b]$ and if $x_0, x_1, \dots, x_n$ are distinct points in $[a,b]$, then there exists a point ξ in $[a,b]$ such that

$$f[x_0, x_1, \dots, x_n] = \frac{1}{n!} f^{(n)}(\xi).$$

Proof. First, let p be the polynomial of degree at most $n-1$ that interpolates f at the nodes $x_0, \dots, x_{n-1}$. Then

$$f(x_n) - p(x_n) = \frac{1}{n!} f^{n}(\xi) \prod_{j=0}^{n-1} (x_n - x_j)$$

From the previous theorem, we have

$$f(x_n) - p(x_n) = f[x_0, \dots, x_n] \prod_{j=0}^{n-1} (x_n - x_j)$$

Combine these two results to arrive at the conclusion.

Hermite - Genocchi formula

Define the simplex in $\mathbb{R}^{n+1}$ by

$$S_n = \left\{ u = (u_0, u_1, \dots, u_n) \in \mathbb{R}^{n+1} : u_i \geq 0, \sum_{i=0}^{n} u_i = 1 \right\}$$

The Hermite-Genocchi formula reads:

$$f[x_0, x_1, \dots, x_n] = \int_{S_n} f^{(n)} (u_0 x_0 + \dots + u_n x_n)\, du$$

Consider the case $n = 1$: Here, we have

$$S_1 = \{ u = (u_0, u_1) \in \mathbb{R}^2 : u_0 \geq 0, u_1 \geq 0, u_0 + u_1 = 1 \}$$

$$= \{ (1-u_1, u_1) : 0 \leq u_1 \leq 1 \}$$

thus:

$$\int_{S_1} f^{(1)} (u_0 x_0 + u_1 x_1)\, du = \int_0^1 f^{(1)} ((1-u_1) x_0 + u_1 x_1)\, du_1$$

$$= \int_0^1 f^{(1)} (x_0 + u_1 (x_1 - x_0))\, du_1$$

$$= \int_0^1 \frac{d}{du_1} f(x_0 + u_1 (x_1 - x_0)) \frac{du_1}{x_1 - x_0}.$$

$$= \frac{1}{x_1 - x_0} \quad f(x_0 + u_1(x_1 - x_0)) \Big|_{u_1=0}^{u_1=1}$$

$$= \frac{1}{x_1 - x_0} \left(f(x_1) - f(x_0) \right)$$

$$= f[x_0, x_1]$$

Inductive step: Assume that the formula is correct for $n-1$. Define

$$I(x_0, x_1, \ldots, x_n) = \int_{S_n} f^{(n)}(u_0 x_0 + \cdots + u_n x_n)\, du$$

Since $\sum_{i=0}^{n} u_i = 1$, $u_0 = 1 - \sum_{i=1}^{n} u_i$.

Therefore,

$$I(x_0, x_1, \ldots, x_n) = \int_{S_n} f^{(n)}\left(\left(1 - \sum_{i=1}^{n} u_i\right) x_0 + u_1 x_1 + \cdots + u_n x_n \right) du$$

$$= \int_{S_n} f^{(n)}\left(x_0 + u_1(x_1 - x_0) + \cdots + u_n(x_n - x_0) \right) du$$

$$= \int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-u_1-\cdots u_{n-1}} f^{(n)}(x_0 + u_1(x_1 - x_0) + \cdots + u_n(x_n - x_0))\, du_n\, du_{n-1} \cdots du_2\, du_1$$

Consider the innermost integration

$$\int_0^{1-u_1-\cdots-u_{n-1}} \frac{d}{du_n} f^{(n-1)}(x_0 + u_1(x_1 - x_0) + \cdots + u_n(x_n - x_0)) \frac{du_n}{x_n - x_0}$$

$$= \frac{1}{x_n - x_0} \left[f^{(n-1)}(x_0 + u_1(x_1 - x_0) + \cdots + u_n(x_n - x_0)) \right]_{u_n=0}^{u_n = 1-u_1-\cdots u_{n-1}}$$

$$= \frac{1}{x_n - x_0} \left[f^{(n-1)}(x_0 + u_1(x_1 - x_0) + \cdots + u_n(x_n - x_0)) \right|_{u_n = 0}$$

$$= \frac{1}{x_n - x_0} \left[f^{(n-1)}\left(x_0 + \sum_{i=1}^{n-1} u_i(x_i - x_0) + \left(1 - \sum_{i=1}^{n-1} u_i\right)(x_n - x_0)\right) - f^{(n-1)}\left(x_0 + \sum_{i=1}^{n-1} u_i(x_i - x_0)\right) \right]$$

$$= \frac{1}{x_n - x_0} \left[f^{(n-1)}\left(x_n + \sum_{i=1}^{n-1} u_i(x_i - x_n)\right) - f^{(n-1)}\left(x_0 + \sum_{i=1}^{n-1} u_i(x_i - x_0)\right) \right]$$

thus the integral becomes:

$$I(x_0, x_1, \ldots, x_n) = \frac{1}{x_n - x_0} \int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-u_1 \cdots u_{n-2}} \cdots$$

$$\left[f^{(n-1)}\left(x_n + \sum_{i=1}^{n-1} u_i(x_i - x_n)\right) - f^{(n-1)}\left(x_0 + \sum_{i=1}^{n-1} u_i(x_i - x_0)\right) \right]$$

$$= \frac{1}{x_n - x_0} \left(I[x_n, x_1, \ldots, x_{n-1}] - I[x_0, \ldots, x_{n-1}] \right)$$

$$= \frac{1}{x_n - x_0} \left(f[x_n, x_1, \ldots, x_{n-1}] - f[x_0, \ldots, x_{n-1}] \right)$$

$$= f[x_0, x_1, \ldots, x_n].$$ $\square$