Hermite Interpolation: Error formula and divided difference method

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Theorem. Let $x_0, x_1, \ldots, x_n$ be distinct nodes in $[a,b]$ and let $f \in C^{2n+2}[a,b]$. If p is the polynomial of degree at most $2n+1$ such that

$$p(x_i) = f(x_i) \qquad p'(x_i) = f'(x_i) \qquad 0 \le i \le n$$

then corresponding to each x in $[a,b]$, there exists a point ξ in (a,b) such that

$$f(x) - p(x) = \frac{f^{(2n+2)}}{(2n+2)!}(\xi) \prod_{i=0}^{n} (x - x_i)^2$$

Proof. If x is a node then the formula is correct. Now fix a point x which is not a node. Define functions ω and ϕ by

$$\omega(t) = \prod_{i=0}^{n} (t - x_i)^2 \qquad \text{and} \qquad \phi = f - p - \lambda\omega$$

where λ is chosen so that $\phi(x) = 0$ ($\lambda = [f(x) - p(x)]/\omega(x)$). Notice that ϕ has at least $n+2$ zeros in $[a,b]$ namely $x, x_0, \ldots, x_n$. By Rolle's thm ϕ' has at least $n+1$ zeros different from the points just enumerated. In addition ϕ' vanishes at $n+1$ of them nodes also. Therefore ϕ' has $2n+2$ zeros in $[a,b]$. By repeated argument, conclude that $\phi^{(2n+2)}$ has a single zero say ξ in (a,b).

Thus:

$$0 = \phi^{(2n+2)}(\xi) = f^{(2n+2)}(\xi) - p^{(2n+2)}(\xi) - \lambda\omega^{(2n+2)}(\xi)$$

Since p is of degree of $2n+1$, $p^{(2n+2)} = 0$. Since $\omega(t)$ has a leading order term t^{2n+2}, it follows $\omega^{(2n+2)} = (2n+2)!$.

Therefore, with $\lambda = \dfrac{f(x) - p(x)}{\omega(x)}$,

$$0 = f^{(2n+2)}(\xi) - \frac{f(x) - p(x)}{\omega(x)} (2n+2)!$$

$\boxtimes$

Extending Newton's divided difference method to Hermite interpolation.

Consider a simple example in which a quadratic polynomial is sought with the conditions :

$$p(x_0) = c_{00} \qquad p'(x_0) = c_{01} \qquad p(x_1) = c_{10}$$

Since the conditions at x_0 is repeated two times, the row corresponding to x_0 also needs to be repeated two times.

$$\begin{array}{ccccc} x_0 & c_{00} & & & \\ & & * & & \\ x_0 & c_{00} & & \$ & \\ & & \# & & \\ x_1 & c_{10} & & & \end{array}$$

In the place of $*$,

$$* = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) = c_{01}$$

$$\begin{array}{cccc} x_0 & c_{00} & & \\ x_0 & c_{00} & c_{01} & \left\{ \frac{c_{10}-c_{00}}{x_1 - x_0} - c_{01} \right\} / x_1 - x_0 \\ x_1 & c_{10} & \frac{c_{10} - c_{00}}{x_1 - x_0} & \end{array}$$

The polynomial will be

$$p(x) = c_{00} + c_{01}(x - x_0) + \left\{ \frac{c_{10}-c_{00}}{x_1 - x_0} - c_{01} \right\} \Big/ x_1 - x_0 \; (x - x_0)^2 .$$

Example 2. $p(x_i) = f(x_i) \qquad p'(x_i) = f'(x_i) \qquad i = 0, 1$

$$\begin{array}{ccccc} x_0 & f(x_0) & & & \\ & & f[x_0, x_0] & & \\ x_0 & f(x_0) & & *1 \;\; f[x_0, x_0, x_1] & \\ & & f[x_0, x_1] & & *3 \;\; f[x_0, x_0, x_1, x_1] \\ x_1 & f(x_1) & & *2 \;\; f[x_0, x_1, x_1] & \\ & & f[x_1, x_1] & & \\ x_1 & f(x_1) & & & \end{array}$$

$$*1 = \frac{f[x_0, x_0] - f[x_0, x_1]}{x_0 - x_1} = f[x_0, x_0, x_1]$$

$$*2 = \frac{f[x_0, x_1] - f[x_1, x_1]}{x_0 - x_1} = f[x_0, x_1, x_1]$$

$$*2 = \frac{f[x_0,x_1] - f[x_1,x_1]}{x_0 - x_1} = f[x_0,x_1,x_1]$$

$$*3 = \frac{f[x_0,x_0,x_1] - f[x_0,x_1,x_1]}{x_0 - x_1} = f[x_0,x_0,x_1,x_1]$$

$$p(x) = f(x_0) + f[x_0,x_0](x-x_0) + f[x_0,x_0,x_1](x-x_0)^2 + f[x_0,x_0,x_1,x_1](x-x_0)^2(x-x_1)$$

Note that: We have the formula:

$$f[x_0, x_1, \dots, x_k] = \frac{1}{k!} f^{(k)}(\xi)$$

where ξ lies in the smallest interval that contains $x_0, x_1, \dots, x_k$.
If the interval length goes to zero, we have

$$f[x_0, x_0, \dots x_0] = \frac{1}{k!} f^{(k)}(x_0) .$$

Therefore when $k \geq 2$, we need to include the factor $1/k!$ also.

Example. $p(1) = 2$ $\quad p'(1) = 3$ $\quad p(2) = 6$ $\quad p'(2) = 7$ $\quad p''(2) = 8$

1	2	3	1	2	-1
1	2	4	3	1	
2	6	7	4		
2	6	7			
2	6				

$$p(x) = 2 + 3(x-1) + (x-1)^2 + 2(x-1)^2(x-2) - (x-1)^2(x-2)^2.$$

Lagrange Form

Consider the problem that seeks a polynomial of the form

$$p(x_i) = c_{i0} \qquad p'(x_i) = c_{i1} \qquad (0 \leq i \leq n) \qquad (*)$$

We try to find a polynomial of the form:

$$p(x) = \sum_{i=0}^{n} c_{i0} A_i(x) + \sum_{i=0}^{n} c_{i1} B_i(x)$$

where A_i and B_i are polynomials with certain properties. For a polynomial to satisfy (*) it is enough to have

$$A_i(x_j) = \delta_{ij} \qquad B_i(x_j) = 0$$
$$A_i'(x_j) = 0 \qquad B_i'(x_j) = \delta_{ij} \qquad \left.\begin{matrix}\\ \\ \end{matrix}\right\} \quad \forall\, i,j \qquad (\ddagger)$$

then

$$p(x_j) = \sum_{i=0}^{n} c_{i0} A_i(x_j) + \sum_{i=0}^{n} c_{i1} B_i(x_j)$$
$$= c_{j0}$$
$$p'(x_j) = \sum_{i=0}^{n} c_{i0} A_i'(x_j) + \sum_{i=0}^{n} c_{i1} B_i'(x_j) = c_{j1}$$

Hence the conditions are satisfied. We construct A_i and B_i as follows. Set the Lagrange Cardinal functions

$$l_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \qquad 0 \le i \le n$$

Define A_i and B_i as

$$A_i(x) = [1 - 2(x - x_i)\, l_i'(x)]\, l_i^2(x) \qquad 0 \le i \le n$$
$$B_i(x) = (x - x_i)\, l_i^2(x)$$

Exc. Verify A_i and B_i satisfy conditions ‡. Also note that A_i and B_i are of degree $2n+1$. This is consistent with the $2n+2$ conditions given.