Review of Calculus

Definition. 1.1 (Continuous function). Let $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, where Ω is an open set. Then the function f is continuous at $\vec{a} \in \Omega$ if for a given $\varepsilon > 0$, $\exists$ a corresponding $\delta > 0$ such that

whenever $\|\vec{x} - \vec{a}\|_{\mathbb{R}^n} < \delta$ and $\vec{x} \in \Omega$

then $\|f(\vec{x}) - f(\vec{a})\|_{\mathbb{R}^m} < \varepsilon$.

Remark. 1.2 If f is continuous at all $\vec{x} \in \Omega$, then f is said to be continuous on Ω.

Remark 1.3. $\|\cdot\|_{\mathbb{R}^d}$ is a norm $\mathbb{R}^d$. Several norms are possible in $\mathbb{R}^d$. We shall often work with the Euclidean norm defined by

$$\|\vec{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

where $\vec{x} = (x_1, x_2, \cdots, x_n)$. Note that all norms in $\mathbb{R}^d$ are equivalent. i.e, $\exists$ constants C_1 and C_2 such that

$$C_1 \|\cdot\|_A \leq \|\cdot\|_B \leq C_2 \|\cdot\|_A$$

for every pair of norms $\|\cdot\|_A$ and $\|\cdot\|_B$ on $\mathbb{R}^d$. The constants C_1 and C_2 are independent of the points in $\mathbb{R}^d$. Therefore therefore continuity wrt one norm implies continuity wrt all other norms.

Theorem 1.4 (Sequential Continuity) A function $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, where Ω is an open set is continuous at $\vec{a} \in \Omega$, iff for any sequence $(\vec{x}_n)_{n \geq 1}$ such that $\vec{x}_n \to \vec{a}$, $f(\vec{x}_n) \to f(\vec{a})$.

Proof. Assume that $(\vec{x}_n)_{n \geq 1} \subseteq \Omega$ and $\vec{x}_n \to \vec{a}$, and f is continuous at $\vec{a}$. Continuity assures that for any $\varepsilon > 0$, $\exists$ corresponding $\delta > 0$ such that

$$\|\vec{x} - \vec{a}\| < \delta \Rightarrow \|f(\vec{x}) - f(\vec{a})\| < \varepsilon.$$

Choose $\varepsilon = 1/n$ for $n \in \mathbb{N}$. Then, $\exists$ $\delta(n)$ such that

$$\|\vec{x} - \vec{a}\| < \delta(n) \Rightarrow \|f(\vec{x}) - f(\vec{a})\| < \frac{1}{n}$$

Since $\vec{x}_n \to \vec{a}$, $\exists$ $N_{\delta(n)} \in \mathbb{N}$ such that

$$n \geq N_{\delta(n)} \Rightarrow \| \vec{x}_n - \vec{a} \| < \delta(n)$$

Then for this $N_{\delta(n)}$,

$$n \geq N_{\delta(n)} \Rightarrow \| \vec{x}_n - \vec{a} \| < \delta(n) \Rightarrow \| f(\vec{x}_n) - f(\vec{a}) \| < 1/n$$

This implies $f(\vec{x}_n) \to f(\vec{a})$.

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Assume that $\vec{x}_n \to \vec{a} \Rightarrow f(x_n) \to f(\vec{a})$. If possible, suppose that f is not continuous at $\vec{a}$. That is there is some $\hat{\varepsilon} > 0$ such that

$$\| f(\vec{x}) - f(\vec{a}) \| > \hat{\varepsilon} \quad \text{for some } \hat{x} \text{ with}$$
$$\| \vec{x} - \vec{a} \| < \delta \quad \text{for all } \delta > 0.$$

Choose $\delta = 1/n$, $n \in \mathbb{N}$. Then $\exists$ $\vec{x}_n$ such that
$\| \vec{x}_n - \vec{a} \| < 1/n$ and $\| f(\vec{x}_n) - f(a) \| > \hat{\varepsilon}$

Thus $(x_n)_{n \geq 1}$ satisfies $\vec{x}_n \to \vec{a}$ yet $f(\vec{x}) \not\to f(\vec{a})$.

This contradiction implies f should be continuous at $\vec{a}$.

Differentiability

Definition 1.5. Suppose that $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a function, where Ω is an open set. The function is said to be differentiable at a point $\vec{x} \in \Omega$, if $\exists$ a linear transformation

$$Df(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^m$$

such that
$$\lim_{|\vec{h}| \to 0} \frac{\| f(\vec{x} + \vec{h}) - f(\vec{x}) - Df(\vec{x})(\vec{h}) \|}{\| \vec{h} \|} = 0$$

Explanation. Suppose that $f : (a, b) \to \mathbb{R}$

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$$f(c+h) = f(c) + f'(c)h + \varepsilon(h)$$

$f : \mathbb{R}^n \to \mathbb{R}^m$

$$f'(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h) - f(\vec{a})}{\vec{h}} \quad \times$$

$$f(\vec{a} + \vec{h}) = f(\vec{a}) + Df(\vec{a})(\vec{h})$$

$$f(c+h) = f(c) + f'(c)h + \varepsilon(h)$$
where $\frac{\varepsilon(h)}{h} \to 0$ as $h \to 0$

$$\left[\frac{f(c+h)-f(c)}{h} = f'(c) + \frac{\varepsilon(h)}{h} \right]$$

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + Df(\vec{a})(\vec{h}) + \varepsilon(h)$$
Such that
$$\lim_{\vec{h}\to 0} \frac{\|\vec{\varepsilon}(h)\|}{\|\vec{h}\|} = 0.$$

Remark. The map $Df(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^m$ can be represented as a matrix of size $m \times n$ WRT the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. Hence, if $f(\vec{x})$ and $\vec{x}$ are represented using the coordinates WRT the standard bases, $Df(\vec{x})$ is an $m \times n$ matrix.

In particular, if $m=1$, then $Df(\vec{x})$ is a $1\times n$ row vector. called the gradient of f at $\vec{x}$ denoted classically by $\nabla f(\vec{x})$.
If m and n are unity, then $Df(x)$ is simply $f'(x)$.

Partial derivative (Rudin)

Consider a function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, where Ω is an open subset. Suppose that $\{\vec{e}_1, \ldots, \vec{e}_n\}$ and $\{\vec{u}_1, \ldots, \vec{u}_m\}$ are standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. Components of f are denoted by $f_1, \ldots, f_m$; this means
$$f(\vec{x}) = \sum_{i=1}^{m} f_i(\vec{x})\vec{u}_i \qquad (\vec{x} \in \Omega),$$
or equivalently by
$$f_i(x) = f(\vec{x}) \cdot \vec{u}_i, \quad 1 \le i \le m$$

For $\vec{x} \in \Omega$, $1 \le i \le m$, and $1 \le j \le n$, we define
$$\frac{\partial f_i}{\partial x_j}(\vec{x}) = D_j f_i(\vec{x}) = \lim_{t\to 0} \frac{f_i(x+te_j) - f_i(x)}{t}$$
provided the limit exists.

Theorem 1.7. Suppose f maps an open set $\Omega \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$ and f is differentiable at a point $\vec{x} \in \Omega$. Then the partial derivative $D_j f_i$ exist, and

$D_j f_i$ exist, and

$$Df(\vec{x})\, e_j = \sum_{i=1}^{n} (D_j f_i)(\vec{x})\, u_i \qquad (1 \leq j \leq n)$$

Theorem 1.8. If $f(x) \leq g(x)$ for all x in an interval that contains a and the limits of f and g both exists as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

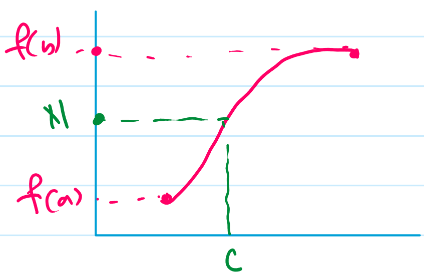
Theorem 1.9 (Sandwitch Thm). Let $f, g,$ and h be given functions such that

1) $f(x) \leq g(x) \leq h(x)$ for all x in an interval containig a, and

2) $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = \ell$,

then $$\lim_{x \to a} g(x) = \ell.$$

Theorem 1.10 (Intermediate Value Thm). Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists $c \in (a, b)$ such that $$f(c) = N.$$



Theorem 1.11 Suppose that f is differentiable at a. Then there exists a function ϕ such that

$$f(x) = f(a) + (x-a) f'(a) + (x-a)\phi(x)$$

and $\lim_{x \to a} \phi(x) = 0$.

Pf. Define $\phi(x) := \dfrac{f(x) - f(a)}{x - a} - f'(a)$

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$$\Rightarrow \lim_{x \to a} \phi(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} - f'(a)$$

$$= f'(a) - f'(a) = 0.$$

Theorem 1.92 (Rolles Thm) Let f be a function that satisfies the following three hypotheses.

1. f is continuous on the closed interval $[a,b]$
2. 〃 diff 〃 open 〃 (a,b)
3. $f(a) = f(b)$

Then $\exists$ c in (a,b) such that $f'(c) = 0$.