Divided difference with repetitions

02 April 2026 12:43

Divided difference with repetitions

Recall that if point $x_0, x_1, \dots, x_n$ are distinct, then $f[x_0, x_1, \dots, x_n]$ has been defined as the coefficient of x^n in the polynomial from Π_n that interpolates f at $x_0, x_1, \dots, x_n$.

We need to define the meaning of interpolation when the list of points contains repetitions.

Definition. We say that f interpolates 0 at $x_1, \dots, x_n$ if $f^{(k-1)}(\xi) = 0$ for each point ξ that occurs k times in the list $x_0, x_1, \dots, x_n$.

Example. We say that f interpolates 0 at $1, 3, 8, 1, 13, 1, 8$ if

$$f(1) = f'(1) = f''(1) = 0$$
$$f(8) = f'(8) = 0$$
$$f(13) = 0. \qquad f(3) = 0.$$

It is easy to verify that (Exc.) a polynomial p interpolates 0 at $x_0, x_1, \dots, x_n$ iff $p(x)$ contains the factor

$$q(x) = \prod_{j=0}^{n} (x - x_j).$$

Remark. If f and g are two functions such that $f-g$ interpolates 0 at $x_0, x_1, \dots, x_n$ then we say that f interpolates g at $x_0, x_1, \dots, x_n$.

Theorem. Let $x_0, x_1, \dots, x_m$ be a list of points in which no element is repeated more than k times. Let f belongs to class C^{k-1} on an interval containing these points. Then there exists a unique polynomial in Π_m that interpolates f at these given points.

Remark. Note that in the general case, $f[x_0, x_1, \dots, x_m]$ is the coefficient of x^n in the polynomial from Π_m that interpolates f at the points $x_0, x_1, \dots, x_m$. If a number ξ occurs k times in this list, then of x in the polynomial from Π_m that interpolates f at the points $x_0, x_1, \ldots, x_m$. If a number ξ occurs k times in this list, then this definition requires the existence of the derivative $f^{(k-1)}(\xi)$. Therefore the general formula is given by

$$p(x) = \sum_{j=0}^{n} f[x_0, x_1, \cdots, x_j] \prod_{i=0}^{j-1} (x - x_i) \qquad (*)$$

with the convention $\prod_{i=0}^{-1} (x - x_i) = 1$.

Theorem. If f is sufficiently differentiable so that the divided difference occuring $(*)$ exists, then the equation $(*)$ gives the polynomial in Π_n that interpolates f at $x_0, x_1, \cdots, x_n$.

Theorem (Recursive). Let $x_0 \leq x_1 \leq \cdots \leq x_n$. Then the divided differences obey the recursive formula:

$$f[x_0, x_1, \cdots, x_n] = \begin{cases} \dfrac{f[x_1, \cdots, x_n] - f[x_0, \cdots, x_{n-1}]}{x_n - x_0} & \text{if } x_n \neq x_0 \\ f^{(n)}(x_0)/n! & \text{if } x_n = x_0. \end{cases}$$

$\boxtimes$