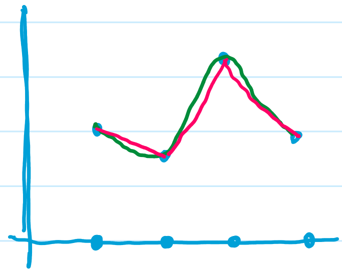
# Spline Interpolation.



Suppose that $n+1$ points denoted by $t_0, t_1, \dots, t_n$ namely <u>knots</u> with $t_0 < t_1 < \dots < t_n$ and an integer $k \geq 0$ is specified. A spline function of degree $k$ having knots $t_0, t_1, \dots, t_n$ is a function $S$ such that

a) On each interval $[t_{i-1}, t_i)$, $S$ is a polynomial of degree $\leq k$.

b) $S$ has a continuous $(k-1)^{st}$ derivative on $[t_0, t_n]$.

Remarks. 1. Splines of degree zero are piecewise constants with the explicit form:

$$S(x) = \begin{cases} S_0(x) = c_0 & x \in [t_0, t_1) \\ S_1(x) = c_1 & x \in [t_1, t_2) \\ \vdots & \\ S_{n-1}(x) = c_{n-1} & x \in [t_{n-1}, t_n] \end{cases}$$

2. Note that the intervals do not intersect, and thus there is no ambiguity in the definition.

A spline of degree 1 is of the form:

$$S(x) = \begin{cases} S_0(x) = a_0 x + b_0 & x \in [t_0, t_1) \\ S_1(x) = a_1 x + b_1 & x \in [t_1, t_2) \\ \vdots & \\ S_{n-1}(x) = a_{n-1} x + b_{n-1} & x \in [t_{n-1}, t_n] \end{cases}$$

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|---|---|---|
| Endsem (Lab) | $25^{th}$ April | |
| Quiz 3 | $22^{nd}$ April | (4 - 4.30 pm) |
| In Lab Exc. | $15^{th}$ April | (10 points) |

Suppose that the values at $t_0, t_1, \dots, t_{n-1}, t_n$ are specified. That is $S(t_i) = y_i$, $0 \leq i \leq n$ is specified. One each interval $[t_i, t_{i+1}]$, the slope of the line connecting $(t_i, y_i)$ and $(t_{i+1}, y_{i+1})$ is given by

$$m_i = \frac{y_{i+1} - y_i}{t_{i+1} - t_i}$$

$$S_i(x) = y_i + m_i(x - t_i)$$

We need to check that $S_i(t_{i+1}) = S_{i+1}(t_{i+1})$ for $0 \le i \le n-2$

$$S_i(t_{i+1}) = y_i + \frac{y_{i+1} - y_i}{t_{i+1} - t_i}(t_{i+1} - t_i) = y_{i+1}$$

$$S_{i+1}(t_{i+1}) = y_{i+1} + \frac{y_{i+2} - y_{i+1}}{t_{i+2} - t_{i+1}}(t_{i+1} - t_{i+1}) = y_{i+1}$$

## Construction of Cubic Splines.

Suppose that each piecewise polynomial is a cubic. The polynomials $S_{i-1}$ and $S_i$ interpolate the same value at the point $t_i$, and therefore

$$S_{i-1}(t_i) = y_i = S_i(t_i) \qquad (1 \le i \le n-1),$$

$S$ is automatically continuous.

Question. Does the continuity of $S, S', S''$ provide enough conditions to define a cubic spline? Each cubic has 4 coefficients, therefore we have $4n$ undetermined coefficients.

- On each subinterval $[t_i, t_{i+1}]$, the two interpolation conditions
$$S(t_i) = y_i, \quad S(t_{i+1}) = y_{i+1}$$
gives $2n$ conditions. The continuity of $S$ gives no additional conditions since it is already been counted in interpolation conditions.

- Continuity of $S', S''$ gives one condition at each interior knot
$$S'_{i-1}(t_i) = S'_i(t_i)$$
$$S''_{i-1}(t_i) = S''_i(t_i)$$
giving $2n-2$ conditions.

In total we have $2n + 2n - 2 = 4n - 2$ condition. The remaining two conditions can be specified at our will. Usually we set

$S''(t_0) = 0 = S''(t_n)$ } Natural Spline.