Cubic spline construction

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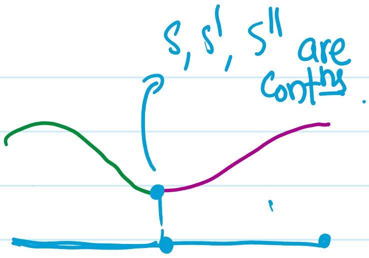

S_i

$t_0 \to S''(t_0) = z_0$

$t_1 \to S''(t_1) = z_1$

$\vdots$

We construct the equation for $S_i(x)$ on each interval $[t_i, t_{i+1}]$. First define

$$z_i = S''(t_i)$$

It follows that $\lim_{x \downarrow t_i} S''(x) = z_i = \lim_{x \uparrow t_i} S''(x)$

$$1 \le i \le n-1$$

This ensures that S'' is continuous at interior knots. Since S_i is a a cubic on $[t_i, t_{i+1}]$, S_i'' is a linear function with

$$S_i''(t_i) = z_i \quad \text{and} \quad S_i''(t_{i+1}) = z_{i+1}.$$

Therefore
$$S_i''(x) = \frac{z_i}{h_i}(t_{i+1} - x) + \frac{z_{i+1}}{h_i}(x - t_i)$$

with $h_i = t_{i+1} - t_i$. Integrate this two times to arrive at

$$S_i(x) = \frac{z_i}{6h_i}(t_{i+1} - x)^3 + \frac{z_{i+1}}{6h_i}(x - t_i)^3 + C(x - t_i) + D(t_{i+1} - x)$$
(Exc.)

with constants of integration C and D. Substitute the interpolation conditions $S_i(t_i) = y_i$ and $S_i(t_{i+1}) = y_{i+1}$ to arrive at

$$S_i(x) = \frac{z_i}{6h_i}(t_{i+1} - x)^3 + \frac{z_{i+1}}{6h_i}(x - t_i)^3 + \left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6}\right)(x - t_i) + \left(\frac{y_i}{h_i} - \frac{z_i h_i}{6}\right)(t_{i+1} - x)$$
(Exc.)

Now the task is to determine $z_1, z_2, \dots, z_{n-1}$. For this we use the continuity of S'. That is

$$S_{i-1}'(t_i) = S_i'(t_i).$$

$$S_i'(t_i) = -\frac{h_i}{\ } z_i - \frac{h_i}{\ } z_{i+1} - y_i/h + y_{i+1}/h.$$

$$S_i'(t_i) = -\frac{h_i}{3} z_i - \frac{h_i}{6} z_{i+1} - y_i/h_i + y_{i+1}/h_i$$

$$S_{i-1}'(t_i) = -h_{i-1}/6\, z_{i-1} + h_{i-1}/3\, z_i - \frac{y_{i-1}}{h_{i-1}} + \frac{y_i}{h_{i-1}}$$

This gives the equation

$$h_{i-1} z_{i-1} + 2(h_i + h_{i-1}) z_i + h_i z_{i+1} = \frac{6}{h_i}(y_{i+1} - y_i) - \frac{6}{h_{i-1}}(y_i - y_{i-1})$$

for $1 \le i \le n-1$. This gives rise to $n-1$ linear equations in $n+1$ unknowns $z_0, z_1, \ldots, z_n$. We can set z_0 and z_n arbitrarily.

The most natural choice is $z_0 = z_n = 0$. This is called a natural spline. The resultant system reads:

$$\begin{bmatrix} u_1 & h_1 & & & & \\ h_1 & u_2 & h_2 & & & \\ & h_2 & u_3 & h_3 & & \\ & & & \ddots & & \\ & & & h_{n-3} & u_{n-2} & h_{n-2} \\ & & & & h_{n-2} & u_{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ \vdots \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_{n-1} \end{bmatrix}$$

$$h_i = t_{i+1} - t_i \qquad u_i = 2(h_i + h_{i-1})$$

$$v_i = b_i - b_{i-1} \quad \text{with} \quad b_i = 6/h_i\,(y_{i+1} - y_i).$$