Taylor's Theorem

14 January 2026 20:05

Taylor's Theorem

Taylor's theorem allows us to approximate generic functions using polynomials with an apriori known bounded error.

Definition 1.1 (Taylor's polynomial for a function at a point)

Let f be n times differentiable at a given point a. The Taylor's polynomial of degree n for the function f at the point a, denoted by T_n, is defined by

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k, \quad x \in \mathbb{R}$$

Theorem 1.2. Let f be an $(n+1)$-times continuously differentiable function on an open interval containing the points a and x. Then there exists a number ξ_x between a and x such that

$$f(x) = T_n(x) + \underbrace{\frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-a)^{n+1}}_{\text{Remainder term}},$$

where T_n is the Taylor's polynomial of degree n for f at the point a given by Def. 1.1 and the second term on RHS is called the remainder term.

Example:

Consider $f(a+h)$:

$$f(a+h) \approx \underbrace{f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \dots + \frac{f^n(a)}{n!}h^n}_{T_n(x)}$$

$$R_n(x) = \frac{f^{n+1}(\xi_h)}{(n+1)!}h^{n+1} \qquad \xi_h \text{ is a point b/w } a \text{ and } a+h.$$

Estimate for the remainder term

Let f be an $(n+1)$ times continuously differentiable function with the property that there exists an M_{n+1} such that

$$|f^{(n+1)}(\xi)| \le M_{n+1} \quad \text{for all } \xi \in I.$$

Then, we have the estimate:

$$\left| \frac{f^{n+1}(\xi)}{(n+1)!} (x-a)^{n+1} \right| \leq \frac{M_{n+1}}{(n+1)!} |x-a|^{n+1}.$$

If the interval is $I = (a, b)$, then further $\longrightarrow \leq \frac{M_{n+1}}{(n+1)!} (b-a)^{n+1}$

This is called the remainder estimate. The remainder error is also called truncation error.

Definition 1.3 (Taylor's series). Let f be C^∞ in a neighborhood of a point a. Then the power series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

is called the Taylor's series of f about the point a.

Theorem 1.4 Let f be $C^\infty(I)$ and let $a \in I$. Assume that there exists an open interval $I_a \subset I$ such that there exists a constant M (may depend on a) such that

$$|f^{(k)}(x)| \leq M^k$$

for all x in I_a and $k \in \mathbb{N} \cup \{0\}$. Then, for each $x \in I_a$, the TS converges and

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k.$$