Order of convergence

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Order of Convergence.

Consider two sequences $a_n = \frac{1}{n}$; $b_n = \frac{1}{n^2}$

Observe that intuitively b_n converges faster than a_n ; we will quantify this using big Oh and little oh notations.

Definition 1.1 (Big Oh and little oh)
Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

① the sequence $\{a_n\}$ is said to be Big Oh of $\{b_n\}$, and write $a_n = O(b_n)$, if there exists a real number C and a natural number N such that

$$|a_n| \leq C\,|b_n| \quad \text{for all } n \geq N.$$

② the sequence $\{a_n\}$ is said to be little oh of b_n and write $a_n = o(b_n)$ if for every $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ such that

$$|a_n| \leq \varepsilon\,|b_n| \quad \forall\, n \geq N$$

Remark: 1) $a_n = O(b_n)$ iff the sequence $\left|\frac{a_n}{b_n}\right|$ is bounded.

2) $a_n = o(b_n)$ iff the sequence $\left|\frac{a_n}{b_n}\right| \to 0$

3) if $a_n = o(b_n)$ then $a_n = O(b_n)$.
But $a_n = O(b_n) \not\Rightarrow a_n = o(b_n)$
example. $a_n = n$, $b_n = n+1$

Rates of convergence

Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} a_n = a$.

1) the rate of convergence is atleast linear if there exists a constant $c < 1$ and a $N \in \mathbb{N}$ such that

$$|a_{n+1} - a| < c\,|a_n - a| \quad \forall\, n \geq N$$

2) The r.o.c is at least quadratic if there exists a Constant C (C not necessarily < 1) and $N \in \mathbb{N}$ such that

$$|a_{n+1} - a| < C\,|a_n - a|^2 \quad \forall\, n \geq N$$

3) The r.o.c is at least α if there exists a constant C (C not necessarily < 1) and $N \in \mathbb{N}$ such that

$$|a_{n+1} - a| < C\,|a_n - a|^{\alpha} \quad \forall\, n \geq N$$