Error in function evaluation

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Total error

Suppose x and y are real nos. Consider $x \odot y$

$$x \odot y - fl(fl(x) \odot fl(y))$$

$$= \underbrace{\{x \odot y - fl(x) \odot fl(y)\}}_{\text{Propagation error}} + \underbrace{\{fl(x) \odot fl(y) - fl(fl(x) \odot fl(y))\}}_{\text{Floating point error}}$$

Propagation of relative error in function evaluation

Consider a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous. Also consider a real no x and its approximation x_A. There are two evaluations $f(x)$ and $f(x_A)$. We are investigating how the relative error $E_r(f(x_A))$ and $E_r(x_A)$ are related.

Assume that f is $\mathcal{C}^1$ fuction.

$$f(x) - f(x_A) = f'(\xi)(x - x_A)$$

where ξ is a point b/w x and x_A.

$$\frac{f(x) - f(x_A)}{f(x)} = \frac{f'(\xi)}{f(x)} \cdot (x - x_A)$$

$$= \frac{f'(\xi)}{f(x)} x \quad \frac{x - x_A}{x}$$

$$E_r(f(x_A)) = \frac{f'(\xi)}{f(x)} x \times E_r(x_A)$$

$$\approx \underbrace{\frac{f'(x)}{f(x)} x}_{\text{Amplification factor.}} \times E_r(x_A).$$

Definition. (Condition number of a function). The condition number of a continuously differentiable function at a point $x = c$ is given by

$$\left| \frac{f'(c)}{f(c)} c \right|$$

The function evaluation is called well-conditioned if the condition number is small. The function is called ill-conditioned if it is not well-conditioned.

Example. Consider the function $f(x) = \sqrt{x}$ for all $x \in [0, \infty)$. Then

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \text{for all } x \in [0, \infty).$$

The condition number is given by

$$CN = \left| \frac{f'(x)}{f(x)} x \right| = \frac{1}{2}$$

$$E_r(f(x_A)) = \frac{1}{2} E_r(x_A)$$

The square root operation is well-conditioned.

Example. $f(x) = {}^{10}/_{1-x^2} \qquad x \in \mathbb{R}$

$$CN = \frac{2x^2}{|1-x^2|} \to \infty \quad \text{as } x \to \pm 1$$

This is an ill-conditioned operation as $x \to \pm 1$.

Remark. However, the condition number is not always a strong indicator of how the relative error is propagated.

$$f(x) = \sqrt{x+1} - \sqrt{x}$$

$$CN = \left| \frac{\frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}}}{\sqrt{x+1} - \sqrt{x}} \times x \right| = \left| \frac{2\sqrt{x} - 2\sqrt{x+1}}{4\sqrt{x(x+1)}} \times x \times \frac{1}{\sqrt{x+1} - \sqrt{x}} \right|$$

$$= \left| \frac{1}{2} \frac{x}{\sqrt{x(x+1)}} \right|$$

$$= \frac{1}{2} \sqrt{\frac{x}{x+1}}$$

For $x \in (0, \infty) \qquad CN \le \frac{1}{2}$

Use a 6-digit arithmetic to compute $f(12345)$

$$f(12345_A) = \sqrt{12346} - \sqrt{12345}$$
$$= 111.113 - 111.108$$
$$= 0.005$$

$$f(12345) = 0.00450032\ldots$$

$$\frac{f(12345) - f(12345_*)}{f(12345)} = -0.111. \qquad 11\% \text{ error}$$

Remark. Thus the condition number is not enough to ensure the accuracy in the corresponding computed value. This introduces the idea of stability.

Stable & unstable computations

Suppose there are n steps to evaluate a f^n at a point c. Then the process of evaluating this function is said to have instability if at least one of the n-steps is ill-conditioned. If all the steps are well conditioned then the process is stable.

Consider $f(x) = \sqrt{x+1} - \sqrt{x}$

$$x_1 = x_0 + 1 \qquad x_2 = \sqrt{x_1} \qquad x_3 = \sqrt{x_0} \qquad x_4 = x_2 - x_3$$

$$\tilde{f}(t) = x_2 - t$$

$$CN = \left| \frac{\tilde{f}'(t)}{\tilde{f}(t)} t \right| = \left| \frac{t}{x_2 - t} \right| \to \infty \text{ as } t \to x_2$$

Thus $\tilde{f}$ is ill-conditioned when $t \to x_2$. Thus the evaluation is unstable.

$$f(x) = \sqrt{x+1} - \sqrt{x} = \frac{1}{\sqrt{x+1} + \sqrt{x}} \quad \Big\} \text{ Rationalisation.}$$

$$x_1 = x_0 + 1 \qquad x_2 = \sqrt{x_1} \qquad x_3 = \sqrt{x_0} \qquad x_4 = x_2 + x_3 \qquad x_5 = \frac{1}{x_4}$$

Exc. ST each of these steps are well-conditioned.
(Thus the evaluation is stable). $\boxtimes$

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