Root finding and bisection method

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Root finding of non-linear equations

In certain biological systems, the growth rate of a population is proportional to the present population at that instance. Suppose that we allow constant immigration with rate b. Then the ODE that models this population growth is given by

$$\frac{dN}{dt} = \underbrace{\lambda N}_{\text{population increase / decrease due to current population}} + \underbrace{b}_{\text{Migration rate}}$$

Then the solution to this ODE is given by

$$N(t) = N_0 e^{\lambda t} + \frac{b}{\lambda}\left(e^{\lambda t} - 1\right)$$

$N(t)$ is the population at time t.
N_0 is the initial population.

Suppose that $N_0 = 1{,}000{,}000$ and $b = 435{,}000$, and $N(1) = 1{,}564{,}000$.
then,

$$1{,}564{,}000 = 1{,}000{,}000\, e^{\lambda} + \frac{435{,}000}{\lambda}\left(e^{\lambda} - 1\right)$$

$$\Rightarrow f(\lambda) = 0 \qquad (*)$$

$$f(\lambda) = 1{,}564{,}000 - 1{,}000{,}000\, e^{\lambda} - \frac{435{,}000}{\lambda}\left(e^{\lambda} - 1\right)$$

$(*)$ is a non linear equation in λ. We need certain root finding algorithms to find the value of λ.

Root finding

The purpose of root finding is to find an approximate solution x^* to the non linear equation

$$f(x) = 0.$$

Note that $f(x^*) \neq 0$, but $f(x^*) \approx 0$.

We assume now onwards that f has an isolated root. That means f has a root $x_R \in (a,b)$ such that f has no other root in (a,b) other than x_R.

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There are two steps:

(a) Initial guess: Establish a small enough interval $[a,b]$ containing one and only one root f. Then take $x_0 \in [a,b]$ as the initial guess.

(b) Improving the value of the root. If this x_0 is not in desired accuracy, then devise a method to improve the accuracy.

Iterative process

x_0 – initial guess

$x_{n+1} = T(x_n)$, where T is the iterative function which changes WRT the method.

Definition (Convergence) A sequence of iterates $(x_n)_{n\geq 1}$ is said to converge with order $p \geq 1$ to a point x_R if

$$|x_{n+1} - x_R| \leq c\,|x_n - x_R|^p \quad , n \geq 0$$

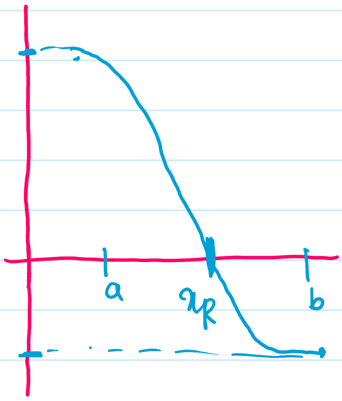
for some constant c.

$p = 1$: Linear convergence
$p = 2$: Quadratic "

Bisection method

Assume that $f : [a,b] \to \mathbb{R}$ is a continuous function such that $f(a) \cdot f(b) < 0$.

Using intermediate value thm, f should have at the least one root in $[a,b]$.



Step 1: $n = 0$: Define $a_0 = a$, $b_0 = b$

Step 2: Define $c_{n+1} = \frac{a_n + b_n}{2}$

Step 3: If $f(a_n)\, f(c_{n+1}) = 0$ $x_R = c_{n+1}$

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If $f(a_n) f(c_{n+1}) < 0$, then $a_{n+1} = a_n$, $b_{n+1} = c_{n+1}$ and root $x_R \in [a_{n+1}, b_{n+1}]$

If $f(a_n) f(c_{n+1}) > 0$, then $a_{n+1} = c_{n+1}$, $b_{n+1} = b_n$ and root $x_R = [a_{n+1}, b_{n+1}]$

Step 4: Stropping criteria: Suppose $\varepsilon > 0$ is given. If

$$|a_{n+1} - b_{n+1}| < \varepsilon \quad \text{Stop the procedure}$$

$$x_A = \frac{a_{n+1} + b_{n+1}}{2}$$

OW go to step 3.

Theorem. Let $[a_0, b_0] = [a, b]$ be the initial interval with $f(a) f(b) < 0$. Define the approximate root as $x_n = (a_{n-1} + b_{n-1})/2$. Then there exists a root x_R such that

$$|x_n - x_R| \leq \frac{1}{2^n}(b-a)$$

Moreover, to achieve the accuracy of $|x_n - x_R| \leq \varepsilon$ it is enough to take

$$n \geq \frac{\log(b-a) - \log \varepsilon}{\log 2}$$

Proof. $b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) \leq \frac{1}{2} \cdot \frac{1}{2}(b_{n-2} - a_{n-2})$

$$\vdots$$

$$\leq \frac{1}{2^n}(b_0 - a_0)$$

$$\therefore \quad |x_n - x_R| \leq \frac{1}{2}(b_{n-1} - a_{n-1}) \leq \frac{1}{2^n}(b_0 - a_0) = \frac{1}{2^n}(b-a).$$

Suppose we need $|x_n - x_R| \leq \varepsilon$. It is enough to take

$$\frac{1}{2^n}(b-a) \leq \varepsilon$$

$\log(b-a) - \log \varepsilon$

$\frac{1}{2^n}(b-a) \leq \varepsilon$

Taking log on both sides $\Rightarrow$ $n \geq \dfrac{\log(b-a) - \log \varepsilon}{\log 2}$.