

## Fixed point iteration method

Many a times, it is possible to recast a root finding problem

$f(x) = 0$  into the form  $g(x) = x$ , which is called a fixed point problem.

Examples.  $f(x) = x^2 - a = 0 \quad a > 0$ . } Root finding problem  
 i)  $x^2 + x - a = x$   
 ii)  $x = a/x$   
 iii)  $x = \frac{1}{2} \left( x + \frac{a}{x} \right)$  } equivalent fixed point problems.

The fixed point iteration method sets an iterative procedure that finds an approximation to the fixed point of the problem. This procedure has two steps

i) Choose an initial guess  $x_0$ .

ii) Define the iteration method by

$$x_{n+1} = g(x_n) \quad n=0, 1, \dots$$

See from the previous examples that the iterative function is not unique. A good iterative function should satisfy three requirements.

- i) For a given starting point  $x_0$ , the successive approximations should be computable.
- ii) The sequence of iterations  $(x_n)_{n \geq 1}$  should converge to some  $\xi$ .
- iii) It should be so that  $g(\xi) = \xi$ .

**Lemma 1.** Let  $g(x)$  be a continuous function on the interval  $a \leq x \leq b$ , and assume that  $a \leq g(x) \leq b$  for every  $a \leq x \leq b$ . (We say  $g$  sends  $[a, b]$  into  $[a, b]$ , and denote it by  $g([a, b]) = [a, b]$ ).

Then  $x = g(x)$  has at least one solution in  $[a, b]$ .

**Proof.** Consider  $f(x) = x - g(x)$   
 $f(a) = a - g(a) \leq 0$  and  $a < g(b) < b$

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 $f(b) = b - g(b) \geq 0$  }  $a \leq g(x) \leq b$

IVT  $\Rightarrow \exists \xi$  such that  $f(\xi) = 0 \Rightarrow \xi - g(\xi) = 0$   
 $\Rightarrow \xi = g(\xi)$   $\square$

Lemma. Let  $g(x)$  be continuous on  $[a, b]$ , and assume that  $g([a, b]) \subset [a, b]$ . Furthermore, assume that there is a constant  $0 < \lambda < 1$ , with

$$|g(x) - g(y)| \leq \lambda |x - y| \quad \forall x, y \in [a, b].$$

Then  $x = g(x)$  has a unique solution  $\alpha$  in  $[a, b]$ . Also the iterates  $x_n = g(x_{n-1})$   $n \geq 1$  will converge to  $\alpha$  for any choice of  $x_0$  in  $[a, b]$  and

$$|\alpha - x_n| \leq \frac{\lambda^n}{1 - \lambda} |x_1 - x_0|. \quad \text{A priori estimate}$$

Proof. Uniqueness: Suppose that there are two solutions  $\alpha$  and  $\beta$ .

$$|\alpha - \beta| = |g(\alpha) - g(\beta)| \leq \lambda |\alpha - \beta|$$

$$\Rightarrow (1 - \lambda) |\alpha - \beta| \leq 0$$

$$\Rightarrow |\alpha - \beta| \leq 0 \quad \Leftrightarrow \alpha = \beta.$$

Convergence: Observe that  $(x_n)_{n \geq 1} \subset [a, b]$  by induction argument and  $g([a, b]) \subset [a, b]$ ,  $x_0 \in [a, b]$ .

$$[x_n \in [a, b] \Rightarrow x_{n+1} = g(x_n) \in [a, b]]$$

$$|\alpha - x_{n+1}| = |g(\alpha) - g(x_n)|$$

$$\leq \lambda |\alpha - x_n| \leq \lambda^2 |\alpha - x_{n-1}|$$

$$\leq \lambda^{n+1} |\alpha - x_0| \quad \forall n \geq 0$$

$$\text{Since } \lambda < 1, \lambda^n \rightarrow 0 \Rightarrow |\alpha - \alpha_{n+1}| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\Rightarrow \alpha_n \rightarrow \alpha$$

Convergence rate:  $|\alpha - \alpha_0| = |\alpha - \alpha_1 + \alpha_1 - \alpha_0|$

$$\leq |\alpha - \alpha_1| + |\alpha_1 - \alpha_0|$$

$$\leq \lambda |\alpha - \alpha_0| + |\alpha_1 - \alpha_0|$$

$$\Rightarrow (1 - \lambda) |\alpha - \alpha_0| \leq |\alpha_1 - \alpha_0|$$

$$\Rightarrow |\alpha - \alpha_0| \leq \frac{1}{1 - \lambda} |\alpha_1 - \alpha_0|$$

$$|\alpha - \alpha_n| \leq \lambda^n |\alpha - \alpha_0|$$

$$\leq \frac{\lambda^n}{1 - \lambda} |\alpha_1 - \alpha_0|$$

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linear rate:  $\lim_{n \rightarrow \infty} \frac{|\alpha - \alpha_{n+1}|}{|\alpha - \alpha_n|} \leq \lambda$  } The sequence of iterates are linearly convergent with a rate at most  $\lambda$ .

A posteriori error estimate:

$$|\alpha - \alpha_n| = |\alpha - \alpha_{n+1} + \alpha_{n+1} - \alpha_n| \leq |\alpha - \alpha_{n+1}| + |\alpha_{n+1} - \alpha_n|$$

$$\leq \lambda |\alpha - \alpha_n| + |\alpha_{n+1} - \alpha_n|$$

$$\Rightarrow (1 - \lambda) |\alpha - \alpha_n| \leq |\alpha_{n+1} - \alpha_n|$$

$$|\alpha - \alpha_{n+1}| \leq \lambda |\alpha - \alpha_n|$$

$$\leq \frac{\lambda}{1 - \lambda} |\alpha_{n+1} - \alpha_n|$$

Assume that  $g(x)$  is differentiable on  $[a, b]$ , then

$$g(x) - g(y) = g'(\xi) (x - y) \text{ for some } \xi \text{ b/w } x \text{ and } y.$$

$$\Rightarrow |g(x) - g(y)| = |g'(\xi)| |x - y|$$

$$\leq \max_{\xi \in [a, b]} |g'(\xi)| |x - y|$$

We define  $\lambda = \max_{a \leq x \leq b} |g'(x)|$

Theorem. Assume that  $g(x)$  is continuously differentiable on  $[a, b]$ , that  $g([a, b]) \subset [a, b]$ , and that

$$\lambda = \max_{a \leq x \leq b} |g'(x)| < 1$$

Then i)  $x = g(x)$  has a unique solution in  $[a, b]$

ii) For any choice of  $x_0$  in  $[a, b]$ , with  $x_{n+1} = g(x_n)$

$$n \geq 0, \quad \lim_{n \rightarrow \infty} x_n = \alpha.$$

$$\text{iii) } |\alpha - x_n| \leq \lambda^n |\alpha - x_0| \leq \frac{\lambda^n}{1-\lambda} |x_1 - x_0|$$

$$\lim_{n \rightarrow \infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha).$$

$$\text{Pf. } \alpha - x_{n+1} = g(\alpha) - g(x_n)$$

$$= g'(\xi_n) (\alpha - x_n) \quad \xi_n \text{ in b/w } \alpha \text{ and } x_n$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = \lim_{n \rightarrow \infty} g'(\xi_n)$$

$$= g'(\alpha) \quad (\text{since } x_n \rightarrow \alpha \text{ as } n \rightarrow \infty)$$

If  $g'(\alpha) \neq 0$ , this method has linear rate of convergence.