Fixed point iteration

03 February 2026 11:55

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Fixed point iteration method

Many a times, it is possible to recast a root finding problem $f(x)=0$ into the form $g(x)=x$, which is called a fixed point problem.

Examples. $f(x) = x^2 - a = 0 \quad a>0.$ } Root finding problem

i) $x^2 + x - a = x$

ii) $x = a/x$

iii) $x = \frac{1}{2}\left(x + \frac{a}{x}\right)$

} equivalent fixed point problems.

The fixed point iteration method sets an iterative procedure that finds an approximation to the fixed point of the problem. This procedure has two steps

i) Choose an initial guess x_0.

ii) Define the iteration method by

$$x_{n+1} = g(x_n) \qquad n=0,1\ldots$$

See from the previous examples that the iterative function is not unique. A good iterative function should satisfy three requirements.

i) For a given starting point x_0, the successive approximates should be computable.

ii) The sequence of iterates $(x_n)_{n\geq 1}$ should converge to some ξ.

iii) It should be so that $g(\xi) = \xi$.

Lemma 1. Let $g(x)$ be a continuous function on the interval $a \leq x \leq b$, and assume that $a \leq g(x) \leq b$ for every $a \leq x \leq b$. (We say g sends $[a,b]$ into $[a,b]$, and denote it by $g([a,b]) = [a,b]$).

Then $x = g(x)$ has at least one solution in $[a,b]$.

Proof. Consider $f(x) = x - g(x)$

$f(a) = a - g(a) \leq 0$ } $a \leq g(x) \leq b$

Proof. Consider $f(x) = x - g(x)$

$$\left.\begin{array}{l} f(a) = a - g(a) \leq 0 \\ f(b) = b - g(b) \geq 0 \end{array}\right\} \quad a \leq g(x) \leq b$$

IVT $\Rightarrow$ $\exists\, \xi$ such that $f(\xi) = 0 \Rightarrow \xi - g(\xi) = 0$
$\Rightarrow \xi = g(\xi)$ $\boxtimes$

Lemma. Let $g(x)$ be continuous on $[a,b]$, and assume that $g([a,b]) \subset [a,b]$. Furthermore, assume that there is a constant $0 < \lambda < 1$, with

$$|g(x) - g(y)| \leq \lambda |x - y| \quad \forall\, x, y \in [a,b].$$

Then $x = g(x)$ has a unique solution α in $[a,b]$. Also the iterates $x_n = g(x_{n-1})$ $n \geq 1$ will converge to α for any choice of x_0 in $[a,b]$ and

$$|\alpha - x_n| \leq \frac{\lambda^n}{1-\lambda} |x_1 - x_0|. \quad \text{A priori estimate}$$

Proof. Uniqueness: Suppose that there are two solutions α and β.

$$|\alpha - \beta| = |g(\alpha) - g(\beta)| \leq \lambda |\alpha - \beta|$$

$$\Rightarrow (1-\lambda)|\alpha - \beta| \leq 0$$

$$\Rightarrow |\alpha - \beta| \leq 0 \iff \alpha = \beta.$$

Convergence: Observe that $(x_n)_{n \geq 1} \subset [a,b]$ by induction argument and $g([a,b]) \subset [a,b]$, $x_0 \in [a,b]$.

$$[\; x_n \in [a,b] \Rightarrow x_{n+1} = g(x_n) \in [a,b] \;]$$

$$|\alpha - x_{n+1}| = |g(\alpha) - g(x_n)|$$

$$\leq \lambda |\alpha - x_n| \leq \lambda^2 |\alpha - x_{n-1}|$$

$$\leq \lambda^{n+1} |\alpha - x_0| \quad \forall\, n \geq 0$$

Since $\lambda < 1$, $\lambda^n \to 0 \Rightarrow |\alpha - x_{n+1}| \to 0$ as $n \to \infty$

$\Rightarrow x_n \to \alpha$

Convergence rate:

$$|\alpha - x_0| = |\alpha - x_1 + x_1 - x_0|$$
$$\leq |\alpha - x_1| + |x_1 - x_0|$$
$$\leq \lambda|\alpha - x_0| + |x_1 - x_0|$$

$$\Rightarrow (1-\lambda)|\alpha - x_0| \leq |x_1 - x_0|$$
$$\Rightarrow |\alpha - x_0| \leq \frac{1}{1-\lambda}|x_1 - x_0|$$

$$|\alpha - x_n| \leq \lambda^n |\alpha - x_0|$$
$$\leq \frac{\lambda^n}{1-\lambda}|x_1 - x_0|$$

Linear rate: $\lim\limits_{n\to\infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|} \leq \lambda$ } The sequence of iterates are linearly convergent with a rate at most λ.

A posteriori error estimate.

$$|\alpha - x_n| = |\alpha - x_{n+1} + x_{n+1} - x_n| \leq |\alpha - x_{n+1}| + |x_{n+1} - x_n|$$
$$\leq \lambda|\alpha - x_n| + |x_{n+1} - x_n|$$

$$\Rightarrow (1-\lambda)|\alpha - x_n| \leq |x_{n+1} - x_n|$$

$$|\alpha - x_{n+1}| \leq \lambda|\alpha - x_n|$$
$$\leq \frac{\lambda}{1-\lambda}|x_{n+1} - x_n|$$

Assume that $g(x)$ is differentiable on $[a,b]$, then

$$g(x) - g(y) = g'(\xi)(x-y) \quad \text{for some } \xi \text{ b/w } x \text{ and } y.$$

$$\Rightarrow |g(x) - g(y)| = |g'(\xi)|\,|x-y|$$
$$\leq \underbrace{\max_{\xi\in[a,b]} g'(\xi)}_{\lambda}\,|x-y|$$

We define $\lambda = \max\limits_{a\leq x\leq b}|g'(x)|$

Theorem. Assume that $g(x)$ is continuously differentiable on $[a,b]$, that $g([a,b]) \subset [a,b]$, and that

$$\lambda = \max_{a \le x \le b} |g'(x)| < 1$$

Then i) $x = g(x)$ has a unique solution in $[a,b]$

ii) For any choice of x_0 in $[a,b]$, with $x_{n+1} = g(x_n)$ $n \geq 0$, $\lim_{n\to\infty} x_n = \alpha$.

iii) $$|\alpha - x_n| \le \lambda^n |\alpha - x_0| \le \frac{\lambda^n}{1-\lambda} |x_1 - x_0|$$

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha).$$

Pf. $$\alpha - x_{n+1} = g(\alpha) - g(x_n)$$
$$= g'(\xi_n)(\alpha - x_n) \qquad \xi_n \text{ in b/w } \alpha \text{ and } x_n$$

$$\Rightarrow \lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = \lim_{n\to\infty} g'(\xi_n)$$
$$= g'(\alpha) \quad (\text{since } x_n \to \alpha \text{ as } n \to \infty)$$

If $g'(\alpha) \neq 0$, this method has linear rate of convergence.

Theorem. Assume that α is a solution of the non linear equation $x = g(x)$, and suppose that $g(x)$ is continuously differentiable in some neighbouring interval about α and $|g'(\alpha)| < 1$. Then $x = g(x)$ has a unique solution α in $[a,b]$; for a sufficiently close choice of x_0, the iterates $x_{n+1} = g(x_n)$ $n \geq 0$ satisfy

$$\lim_{n\to\infty} x_n = \alpha$$

and $$|\alpha - x_n| \le \lambda^n |\alpha - x_0| \le \frac{\lambda^n}{1-\lambda} |x_1 - x_0|$$

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha).$$

Proof. It is enough to find an interval $[a_\varepsilon, b_\varepsilon]$ around α such that $g([a_\varepsilon, b_\varepsilon]) \subseteq [a_\varepsilon, b_\varepsilon]$ and

$$\max |g'(x)| < 1.$$

$$\max_{x\in[a_\varepsilon, b_\varepsilon]} |g'(x)| < 1.$$

To achieve this, pick a number λ such that

$$|g'(\alpha)| < \lambda < 1.$$

Then choose an interval $I = [\alpha-\varepsilon, \alpha+\varepsilon]$ with

$$\max_{x\in I} |g'(x)| \le \lambda < 1$$

$x \in [\alpha-\varepsilon, \alpha+\varepsilon]$,

Note that
$$\begin{aligned} |\alpha - g(x)| &= |g(\alpha) - g(x)| \\ &= |g'(\xi)|\,|\alpha - x| \\ &\le |\alpha - x| \le \varepsilon \end{aligned} \Bigg\} \quad |g'(\xi)| \le \lambda < 1 \text{ on } [\alpha-\varepsilon, \alpha+\varepsilon]$$

$$\Rightarrow \quad g(x) \in [\alpha-\varepsilon, \alpha+\varepsilon]$$

$$\Rightarrow \quad g: [\alpha-\varepsilon, \alpha+\varepsilon] \longrightarrow [\alpha-\varepsilon, \alpha+\varepsilon].$$

Now apply the previous thm with the interval $[a,b] = [\alpha-\varepsilon, \alpha+\varepsilon]$.

Example. Consider the fixed point scheme $x = x + c(x^2-3)$, pick c to ensure convergence. Note that the fixed point of the function
$$g(x) = x + c(x^2-3) \text{ is } \sqrt{3}$$

$$g'(x) = 1 + 2cx$$

We need $\quad -1 < g'(\sqrt{3}) < 1 \iff -1 < 1 + 2c\sqrt{3} < 1$

A possible choice is $\quad 1 + 2c\sqrt{3} = 0 \Rightarrow c = -1/2\sqrt{3} \approx -1/4$

<u>Higher-Order Convergence</u>

Theorem. Assume that α is a root of $x = g(x)$, and that $g(x)$ is p-times continuously differentiable for all x near to α, for $p \geq 2$. Furthermore assume that

$$g'(\alpha) = \cdots = g^{(p-1)}(\alpha) = 0$$

Then if the initial guess x_0 is chosen sufficiently close to α, the

Then if the initial guess x_0 is chosen sufficiently close to α, the iteration

$$x_{n+1} = g(x_n) \quad n \geq 0$$

will have order of convergence p, and

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{(\alpha - x_n)^p} = (-1)^{p-1} \frac{g^{(p)}(\alpha)}{p!}$$

Proof. Expand $g(x_n)$ about α

$$x_{n+1} = g(x_n) = g(\alpha) + (x_n - \alpha) g'(\alpha) + \cdots + \frac{(x_n-\alpha)^{p-1}}{(p-1)!} g^{(p-1)}(\alpha)$$

$$+ \frac{(x_n-\alpha)^p}{p!} g^{(p)}(\xi_n),$$

where ξ_n is a point between x_n and α.

$$\Rightarrow x_{n+1} = \alpha + \frac{(x_n-\alpha)^p}{p!} g^{(p)}(\xi_n)$$

$$\Rightarrow x_{n+1} - \alpha = \frac{(x_n-\alpha)^p}{p!} g^{(p)}(\xi_n)$$

$$\Rightarrow \frac{\alpha - x_{n+1}}{(\alpha - x_n)^p} = (-1)^{p-1} \frac{g^{(p)}}{p!} (\xi_n)$$

$$\Rightarrow \lim_{n\to\infty} \frac{\alpha - x_{n+1}}{(\alpha - x_n)^p} = \frac{(-1)^{p-1}}{p!} g^{(p)}(\alpha).$$