Newton's method

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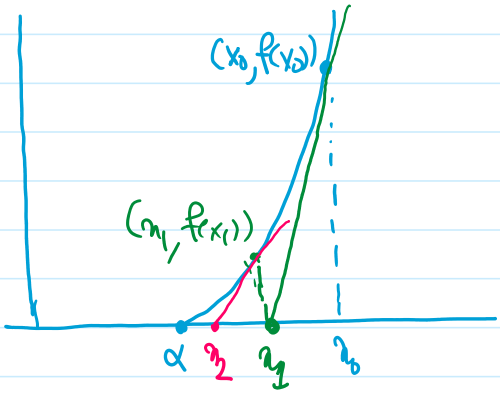
Newtons method attempts to provide an approximate root of $f(x)=0$. Starting from an initial guess x_0. There are two ways of deriving Newton's method.

Graphical method.



Observe that the x-intercept of the tangent line to $y=f(x)$ at x_0 $(= x_1)$ is closer approximation to α than x_0. The equation of the tangent line is given by

$$y = f'(x_0)(x-x_0) + f(x_0)$$

Then, the x-intercept is given by $y=0$

$$f'(x_0)(x_1-x_0) + f(x_0) = 0$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Repeat this procedure to get the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n \geq 0.$$

Analytical method

Suppose that α is the root. Then the Taylor polynomial of $f(x)$ about $x = x_n$ at $x=\alpha$ is given by

$$f(\alpha) = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha-x_n)^2}{2} f''(\xi_n)$$

for some ξ_n between α and x_n. Since $f(\alpha)=0$

$$0 = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(\xi_n)$$

$$\alpha = x_n - \frac{f(x_n)}{f'(x_n)} - (\alpha - x_n)^2 \frac{f''(\xi_n)}{2f'(x_n)}$$

If we avoid the error term in the RHS, Then we obtain

$$\alpha \approx x_n - \frac{f(x_n)}{f'(x_n)} = x_{n+1}$$

Here, x_{n+1} serve as a better approximation to α than x_n. More over the error term is given by

$$\alpha - x_{n+1} = -(\alpha - x_n)^2 \frac{f''(\xi_n)}{2f'(x_n)} \qquad n \geq 0$$

This suggests that Newton's method is quadratically convergent.

Theorem. Assume that $f(x)$, $f'(x)$, and $f''(x)$ are continuous for all x in some neighbourhood of α, and assume that $f(\alpha)=0$, $f'(\alpha) \neq 0$. Then if x_0 is sufficiently close to α, the iterates x_n, $n \geq 0$ with

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

will converge to α. Moreover

$$\lim_{n \to \infty} \frac{\alpha - x_{n+1}}{(\alpha - x_n)^2} = -\frac{f''(\alpha)}{2f'(\alpha)}$$