Newton's method

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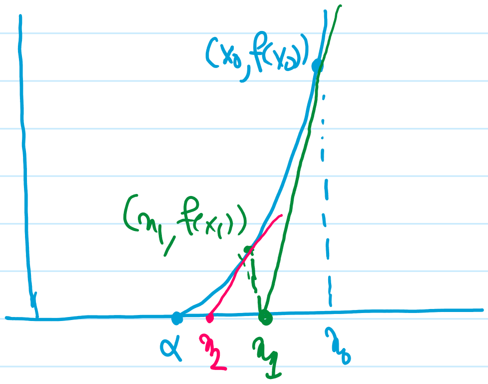
Newtons method attempts to provide an approximate root of $f(x)=0$. starting from an initial guess x_0. There are two ways of deriving Newton's method.

Graphical method.



Observe that the x-intercept of the tangent line to $y=f(x)$ at x_0 ($=x_1$) is closer approximation to α than x_0. The equation of the tangent line is given by

$$y = f'(x_0)(x-x_0) + f(x_0)$$

Then, the x-intercept is given by $y=0$

$$f'(x_0)(x_1-x_0) + f(x_0) = 0$$

$$\Longrightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Repeat this procedure to get the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad n \geq 0.$$

Analytical method

Suppose that α is the root. Then the Taylor polynomial of $f(x)$ about $x=x_n$ at $x=\alpha$ is given by

$$f(\alpha) = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha-x_n)^2}{2} f''(\xi_n)$$

for some ξ_n between α and x_n. Since $f(\alpha)=0$

$$0 = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(\xi_n)$$

$$\alpha = x_n - \frac{f(x_n)}{f'(x_n)} - (\alpha - x_n)^2 \frac{f''(\xi_n)}{2f'(x_n)}$$

If we avoid the error term in the RHS, Then we obtain

$$\alpha \approx x_n - \frac{f(x_n)}{f'(x_n)} = x_{n+1}$$

Here, x_{n+1} serve as a better approximation to α than x_n. More over the error term is given by

$$\alpha - x_{n+1} = -(\alpha - x_n)^2 \frac{f''(\xi_n)}{2f'(x_n)} \qquad n \geq 0$$

This suggests that Newton's method is quadratically convergent.

Theorem. Assume that $f(x)$, $f'(x)$, and $f''(x)$ are continuous for all x in some neighbourhood of α, and assume that $f(\alpha) = 0$, $f'(\alpha) \neq 0$. Then if x_0 is sufficiently close to α, the iterates x_n, $n \geq 0$ with

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

will converge to α. Moreover

$$\lim_{n \to \infty} \frac{\alpha - x_{n+1}}{(\alpha - x_n)^2} = -\frac{f''(\alpha)}{2f'(\alpha)}$$

Proof. Since $f'(x)$ is continuous and $f'(\alpha) \neq 0$, there exists a small enough interval $I = [\alpha - \varepsilon, \alpha + \varepsilon]$ such that $f'(x) \neq 0$ on I.

Define

$$M = \frac{\max_{x \in I} |f''(x)|}{2 \min_{x \in I} |f'(x)|}$$

Note that $\qquad \alpha - x_{n+1} = -(\alpha - x_n)^2 \dfrac{f''(\xi_n)}{2f'(x_n)} \qquad n \geq 0 \qquad \xi_n$ is a point b/w x_n and α.

$$\Rightarrow \quad \alpha - x_1 = -(\alpha - x_0)^2 \, \frac{f''(\xi_0)}{2f'(x_0)}$$

$$\Rightarrow \quad |\alpha - x_1| = |\alpha - x_0|^2 \times \left| \frac{f''(\xi_0)}{2f'(x_0)} \right|$$

$$\Rightarrow \quad |\alpha - x_1| \leq M |\alpha - x_0|^2$$

$$\Rightarrow \quad M|\alpha - x_1| \leq (M|\alpha - x_0|)^2$$

Choose x_0 such that $M|\alpha - x_0| < 1 \Rightarrow M|\alpha - x_1| < 1$

$$\Rightarrow \quad M|\alpha - x_1| \leq M|\alpha - x_0|$$

$$\Rightarrow \quad |\alpha - x_1| \leq |\alpha - x_0| \leq \varepsilon$$

By induction, $|\alpha - x_n| \leq \varepsilon$ and $M|\alpha - x_n| < 1$ for all $n \geq 1$.

To show the convergence,

$$|\alpha - x_{n+1}| \leq M|\alpha - x_n|^2$$

$$\begin{aligned}
\Rightarrow \quad M|\alpha - x_{n+1}| &\leq (M|\alpha - x_n|)^2 \\
&\leq ((M|\alpha - x_{n-1}|)^2)^2 \\
&= (M|\alpha - x_{n-1}|)^{2^2} \\
&\ \ \vdots \\
&\leq (M|\alpha - x_0|)^{2^{n+1}}
\end{aligned}$$

$$\Rightarrow \quad |\alpha - x_{n+1}| \leq \frac{1}{M}(M|\alpha - x_0|)^{2^{n+1}} \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \quad x_n \to \alpha \quad \text{as } n \to \infty.$$

Thus convergence is proved.

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{(\alpha - x_n)^2} = -\lim_{n\to\infty} \frac{f''(\xi_n)}{2f'(x_n)} = \frac{-f''(\alpha)}{2f'(\alpha)}$$

(since ξ_n is a point between x_n and α) $\boxtimes$

The theorem says that to obtain the convergence the initial guess x_0 should be such that

$|\alpha - x_0| < 1$.

the theorem says that to ensure the convergence the initial guess x_0 should be such that

$$|\alpha - x_0| < \frac{1}{M}.$$