Regula Falsi

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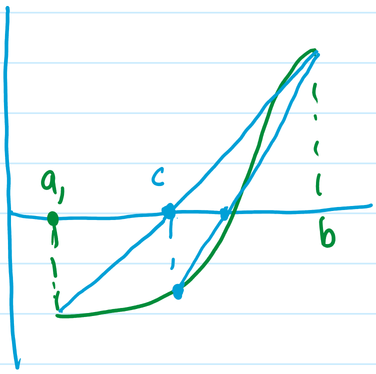
Regula Falsi or Method of false position

Bisection method has the advantage that if $f(a)\,f(b) < 0$, then the iterative sequence converge to a root. However, this convergence is slow under the circumstances

a) if the interval $[a,b]$ is large

b) if the root is near one of the end points.

In regula falsi method, we approximate the function $y=f(x)$, by the straight line joining $(a, f(a))$ and $(b, f(b))$ and the root α is approximated by the x-intercept of this line.



Equation of this line is given by

$$y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

The x-intercept is given by (denoted by x_1)

$$f(a) + \frac{f(b)-f(a)}{b-a}(x_1 - a) = 0$$

$$\Rightarrow x_1 = \frac{a f(b) - b f(a)}{b-a}$$

Now we proceed as in the bisection method:

If $f(x_1) = 0$, then the root $\alpha = x_1$

If $f(a)\cdot f(x_1) < 0$, then $[a_1, b_1] = [a, x_1]$ and repeat

If $f(a)\,f(x_1) > 0$, then $[a_1, b_1] = [x_1, b]$ and repeat

We generalise the procedure as follows:

1. For $n = 0, 1, 2, \ldots$ define the iterative sequence by

$$x_{n+1} = a_n - f(b_n)\frac{b_n - a_n}{f(b_n) - f(a_n)} = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}$$

which is the x-intercept of the line joining $(a_n, f(a_n))$, $(b_n, f(b_n))$

2. If x_{n+1} solves $f(x_{n+1}) = 0$, then that is the root.

3. Define the subinterval $[a_{n+1}, b_{n+1}]$ of $[a_n, b_n]$ as follows:

$$[a_{n+1}, b_{n+1}] = \begin{cases} [a_n, x_{n+1}] & \text{if } f(a_n) f(x_{n+1}) < 0 \\ [x_{n+1}, b_n] & \text{if } f(b_n) f(x_{n+1}) < 0 \end{cases}$$

Bisection method always converge as the length of the intervals always go to zero. However, the length of subintervals in RF method may not go to zero if f is concave or convex in the interval $[a, b]$. Therefore, we cannot specify a stopping criteria in the case of RF method

<u>Convergence analysis of RF method</u>

We obtain two sequences from regula falsi method.

1. $a \leq a_1 \leq a_2 \leq \cdots \leq b$ (bdd above sequence monotonically non-decreasing)

Hence $\lim_{n\to\infty} a_n = \alpha$ exists.

2. $a \leq \cdots \leq b_1 \leq b_0 \leq b$ (bdd below sequence monotonically non-increasing)

Hence $\lim_{n\to\infty} b_n = \beta$ exists.

Since $a_n \leq b_n$, we can also conclude that

$$\alpha = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n = \beta.$$

If the lengths of the interval obtained by the regula-falsi method tend to zero, then we have

$$\alpha = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \beta$$

in which case by sandwich theorem,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} b_n$$

as $a_n \leq x_{n+1} \leq b_n$ for all $n \geq 0$. In this case, the common limit will be a root of the nonlinear equation.

But, sometimes it is possible that, the length of subintervals chosen by regula falsi method do not go to zero. In other words, if $a_n \to \alpha$ and $b_n \to \beta$, then it may that $\alpha \neq \beta$, in fact $\alpha < \beta$.

But it can be proved that $x_n \to r$ with $f(r) = 0$.

Theorem. Let $f: [a_0, b_0] \to \mathbb{R}$ be a continuous function such that $f(a_0)$ and $f(b_0)$ have opposite signs. If the iterates $(x_n)_{n\geq 1}$ is the iterative sequence defined by regula-falsi method, then

$$x_n \to r \in (a_0, b_0)$$

with $f(r) = 0$.

Proof. By the construction of the iterates of the regula-falsi method, it follows that

$$a_n \leq x_n \leq b_n \quad \text{for all } n \geq 1.$$

Therefore $(x_n)_{n\geq 1}$ is a bounded sequence in $[a_0, b_0]$. Hence, by Bolzano-Weirstrass theorem, then there exists a convergent subsequence $(x_{n_k})_{k\geq 1}$ with $x_{n_k} \to r$ as $k \to \infty$.

Wlog assume that $f(a_0) < 0$ and $f(b_0) > 0$. Then we claim that $f(a_n) < 0$ and $f(b_n) > 0$ for all $n \geq 1$. This can be proved by induction. The base case holds as $f(a_0) < 0$ and $f(b_0) > 0$. Assume that $f(a_n) < 0$ and $f(b_n) > 0$. If $f(x_{n+1}) < 0$, $[a_{n+1}, b_{n+1}] = [x_{n+1}, b_n]$, hence $f(a_{n+1}) = f(x_{n+1}) < 0$, $f(b_{n+1}) = f(b_n) > 0$.

If $f(x_{n+1}) > 0$, $[a_{n+1}, b_{n+1}] = [a_n, x_{n+1}]$, hence $f(a_{n+1}) = f(a_n) < 0$

and $f(b_{n+1}) = f(x_{n+1}) > 0$.

Therefore, the sequences $(f(a_n))_{n\geq 1}$ and $(f(b_n))_{n\geq 1}$ preserve their sign.

Since $a_{n_k} \leq x_{n_k} \leq b_{n_k}$, and $a_{n_k} \to \alpha$ and $b_{n_k} \to \beta$,

it follows that $\alpha \leq r \leq \beta$. Since one of the sequences (a_{n_k}) or (b_{n_k}) should contain infinitely many terms of (x_{n_k}),

it follows that either $\alpha = r$ or $\beta = r$. wlog assume that $\beta = r$.

Then $$f(r) = f(\beta) = f(\lim_{k\to\infty} b_{n_k})$$
$$= \lim_{k\to\infty} f(b_{n_k}) \geq 0.$$

Suppose that if possible $f(\beta) > 0$. Then there exists a neighbourhood $[\beta-\delta, \beta+\delta]$ of β such that $f(x) > 0$ on $[\beta-\delta, \beta+\delta]$. Since

$x_{n_k} \to \beta$ as $k \to \infty$, $\exists\ K \in \mathbb{N}$

$$\beta - \delta \leq x_{n_k} \leq \beta + \delta \quad \forall\ k \geq K$$

Since infinitely many terms of (x_{n_k}) is same as that of (b_{n_k})

$\exists\ \hat{k} > K$ such that $x_{n_{\hat{k}}} = b_{n_{\hat{k}}}$. By monotonicity of (b_{n_k}) it follows that

$$\beta < \cdots \leq b_{n_{\hat{k}+1}} \leq b_{n_{\hat{k}}} \leq \beta + \delta$$

since $f(b_{n_k}) > 0$, for all $k \geq \hat{k}$, $f(a_{n_k}) < 0$, leading to $a_n = a_{n_{\hat{k}}}$

for all $n \geq \hat{k}$. and $x_n = b_n$. Therefore, the regula-falsi method

$$x_{n+1} = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}$$

implies

$$x_{n+1} = \frac{a_{n_{\hat{k}}} f(b_n) - b_n f(a_{n_{\hat{k}}})}{f(b_n) - f(a_{n_{\hat{k}}})} \quad \forall\ n \geq n_{\hat{k}}$$

Taking lim $n \to \infty$ implies,

$$r = \frac{a_{n_{\hat{k}}} f(r) - r f(a_{n_{\hat{k}}})}{f(r) - f(a_{n_{\hat{k}}})}$$

$$\Leftrightarrow \quad (r - a_{n_{\hat{k}}}) f(r) = 0$$

Since $r \neq a_{n_{\hat{k}}}$ (why?) it follows that $f(r) = 0$, which contradicts our assumption. Hence $x_n \to r$ and $f(r) = 0$.